中山大學 SUN YAT-SEN UNIVERSITY

Heavy double-gluon hybrid mesons with exotic quantum numbers in QCD sum rules

Ding-Kun Lian

Email: liandk@mail2.sysu.edu.cn

Sun Yat-Sen University

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Collaborators: Q. N. Wang, X. L. Chen, P. F. Yang, W. Chen and H.X. Chen

51 AND 45 YEARS OF QCD AND SVZ SUM RULES
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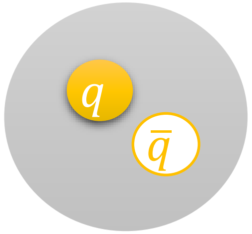
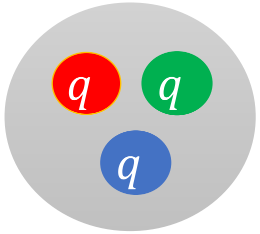
- Current operators for double-gluon hybrid mesons
- QCD sum rules and results
- Conclusion

Background

Conventional quark model:



$\bar{q}q$ (meson)

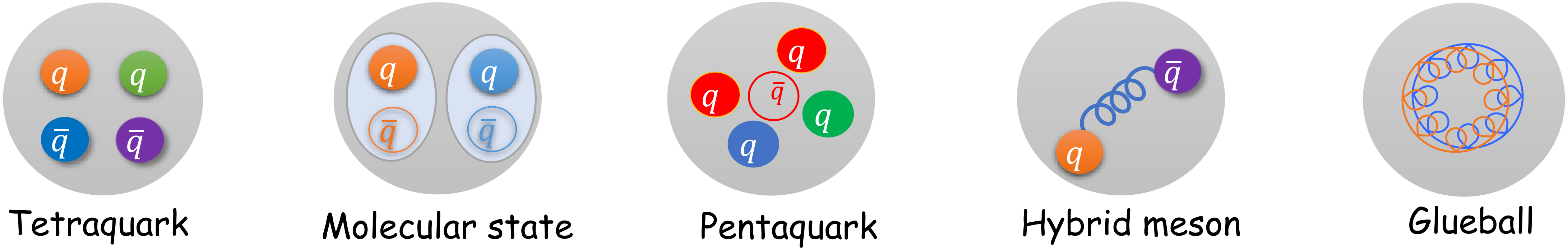


qqq (baryon)

	$S = 0$	$S = 1$
$L = 0$	0^{-+}	1^{--}
$L = 1$	1^{+-}	$(0,1,2)^{++}$
$L = 2$	2^{-+}	$(1,2,3)^{--}$
…	…	…

Quantum numbers for normal mesons

Exotic hadrons: flavor, spin, charge



Exotic quantum number: $\mathbf{0^{--}}$, $\boldsymbol{odd^{-+}}$, $\boldsymbol{even^{+-}}$

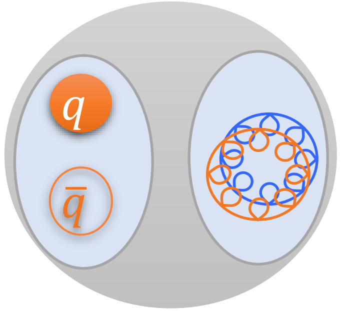
Charge: meson-like with two electric charges, baryon-like with three electric charges

Background

A new hadron configuration: double-gluon hybrid meson ($\bar{q}GGq$) (PRD 105 (2022) 5, L051501)

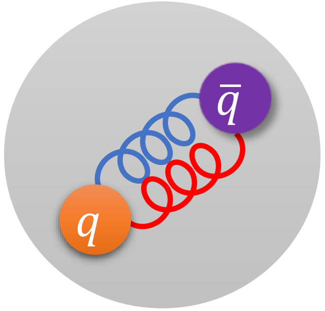
Color structure: $$\left(\bar{\mathbf{3}} \otimes \mathbf{3}\right)_{[\bar{Q}Q]} \otimes (\mathbf{8} \otimes \mathbf{8})_{[GG]} = (\mathbf{1} \oplus \mathbf{8})_{[\bar{Q}Q]} \otimes \left(\mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}\right)_{[GG]}$$
$$= \left(\mathbf{1}_{[\bar{Q}Q]} \otimes \mathbf{1}_{[GG]}\right) \oplus \left(\mathbf{8}_{[\bar{Q}Q]} \otimes \mathbf{8}_{[GG]}\right) \oplus \left(\mathbf{8}_{[\bar{Q}Q]} \otimes \mathbf{8}_{[GG]}\right) \oplus \cdots$$

Three color-singlets



Glueball-meson molecule

(PLB 843, 138030 (2023))



$\times\ 2$

Compact hybrid meson

We will focus on the $\mathbf{8}_{[\bar{Q}Q]} \otimes \mathbf{8}_{[GG]}$ structure.

Background

It is no doubt that one more valence gluon field will enrich the spectra of the hybrid mesons. The heavy double-gluon hybrid mesons with $J^{PC}=1^{--}$ were studied by Tang et al (PRD 105 (2022) 11, 114004) using the vector interpolating currents in QCD sum rules. Su et al (PRD 107 (2023) 3, 034010, PRD 107 (2023) 11, 114005, PRD 109 (2024) 1, L011502) investigated the light and heavy double-gluon hybrid mesons via the current operators with even Lorentz indices.

In this talk, we will discuss the current operators for double-gluon hybrid mesons and study the heavy double-gluon hybrid mesons with exotic quantum numbers $J^{PC}=1^{-+}$ and 2^{+-} using operators distinct from the previous studies.

Current operators

Form of the color-singlet current for the double-gluon hybrid meson:

$$J = \bar{Q}^a (T^r T^s)_{ab} Q^b G^r G^s$$

where $Q = c/b$ represents a charm/bottom quark field, G is the gluon field strength, T's are the generators of the SU(3) group. The Lorentz indices are ignored.

We know that

$$T^r T^s = \frac{1}{2}[\frac{1}{3}\delta_{rs}\mathbf{1} + (d^{rst} + i f^{rst})T^t]$$

So there are three kinds of configurations for double-gluon hybrid mesons:

$$J = \begin{cases} \bar{Q}QGG & \text{Glueball-meson molecule} \\ \bar{Q}T^t Q d^{rst} G^r G^s & \text{Symmetry glueball operator} \\ \bar{Q}T^t Q f^{rst} G^r G^s & \text{Anti-symmetry glueball operator} \end{cases}$$

Current operators

Operators	P	C
$\bar{Q}_a Q_b$	$+$	$+$
$\bar{Q}_a \gamma_5 Q_b$	$-$	$+$
$\bar{Q}_a \gamma_\mu Q_b$	$(-1)^\mu$	$-$
$\bar{Q}_a \gamma_\mu \gamma_5 Q_b$	$-(-1)^\mu$	$+$
$\bar{Q}_a \sigma_{\mu\nu} Q_b$	$(-1)^\mu(-1)^\nu$	$-$
$d^{npq} G_p^{\alpha\beta} G_q^{\gamma\delta} T_n^{ab}$	$(-1)^\alpha(-1)^\beta(-1)^\gamma(-1)^\delta$	$+$
$f^{npq} G_p^{\alpha\beta} G_q^{\gamma\delta} T_n^{ab}$	$(-1)^\alpha(-1)^\beta(-1)^\gamma(-1)^\delta$	$-$
$d^{npq} \tilde{G}_p^{\alpha\beta} G_q^{\gamma\delta} T_n^{ab}$	$-(-1)^\alpha(-1)^\beta(-1)^\gamma(-1)^\delta$	$+$
$f^{npq} \tilde{G}_p^{\alpha\beta} G_q^{\gamma\delta} T_n^{ab}$	$-(-1)^\alpha(-1)^\beta(-1)^\gamma(-1)^\delta$	$-$

Parity and C-parity of the color octet quark-antiquark and double-gluon operators.

Current operators

1^{-+}

$$J^1_\mu = g_s^2 \bar{Q}_i T_a^{ij} \gamma_\rho Q_j f^{abc} G^b_{\mu\nu} G^{c,\nu\rho}$$

$$J^2_\mu = g_s^2 \bar{Q}_i T_a^{ij} \gamma_\rho Q_j f^{abc} \tilde{G}^b_{\mu\nu} \tilde{G}^{c,\nu\rho}$$

$$J^3_\mu = g_s^2 \bar{Q}_i T_a^{ij} \gamma_\mu Q_j f^{abc} G^b_{\alpha\beta} G^{c,\alpha\beta}$$

$$J^4_\mu = g_s^2 \bar{Q}_i T_a^{ij} \gamma_\mu \gamma_5 Q_j d^{abc} \tilde{G}^b_{\alpha\beta} G^{c,\alpha\beta}$$

$$J^5_\mu = g_s^2 \bar{Q}_i T_a^{ij} \gamma_\rho \gamma_5 Q_j d^{abc} \tilde{G}^b_{\mu\nu} G^{c,\nu\rho}$$

$$J^6_\mu = g_s^2 \bar{Q}_i T_a^{ij} \gamma_\rho \gamma_5 Q_j d^{abc} G^b_{\mu\nu} \tilde{G}^{c,\nu\rho}$$

among which J^1_μ and J^2_μ were first studied by Tang et al (PRD 105 (2022) 11, 114004), but the quantum numbers were considered incorrectly as 1^{--}.

However, one can demonstrate that these six currents are not independent due to the symmetries of the glueball operators. They have the following relations,

$$J^1_\mu = -J^2_\mu \qquad J^5_\mu = j^6_\mu = -\frac{1}{4} J^4_\mu \qquad J^3_\mu = 0$$

Current operators

2^{+-}

Two Lorentz indices: $J^1_{\mu\nu} = g_s^2\bar{Q}_i T_a^{ij}\gamma_5 Q_j f^{abc}\tilde{G}^b_{\mu\alpha}G^{c,\alpha\nu}$ $\quad J^2_{\mu\nu} = g_s^2\bar{Q}_i T_a^{ij} Q_j f^{abc}G^b_{\mu\alpha}G^{c,\alpha\nu}$

$$J^3_{\mu\nu} = g_s^2\bar{Q}_i T_a^{ij}\sigma^{\alpha\beta} Q_j d^{abc}G^b_{\mu\alpha}G^c_{\beta\nu}$$

However, $J^2_{\mu\nu} = -J^2_{\nu\mu}$, $J^3_{\mu\nu} = -J^3_{\nu\mu}$. They cannot couple to the state with $J = 2$.

Only $J^1_{\mu\nu}$ can couple to 2^{+-}, but it only contributes to the perturbative term.

Invoking Three Lorentz indices:

$$J^1_{\mu\nu\rho} = g_s^2\bar{Q}_i T_a^{ij}\gamma_\alpha Q_j d^{abc}G^b_{\mu\nu}G^{c,\alpha}{}_\rho(\tilde{G}^b_{\mu\nu}G^{c,\alpha}{}_\rho)$$

$$J^2_{\mu\nu\rho} = g_s^2\bar{Q}_i T_a^{ij}\gamma_\alpha Q_j d^{abc}G^b_{\mu\nu}\tilde{G}^{c,\alpha}{}_\rho(\tilde{G}^b_{\mu\nu}\tilde{G}^{c,\alpha}{}_\rho)$$

$$J^3_{\mu\nu\rho} = g_s^2\bar{Q}_i T_a^{ij}\gamma_\alpha\gamma_5 Q_j f^{abc}G^b_{\mu\nu}G^{c,\alpha}{}_\rho(\tilde{G}^b_{\mu\nu}G^{c,\alpha}{}_\rho)$$

$$J^4_{\mu\nu\rho} = g_s^2\bar{Q}_i T_a^{ij}\gamma_\alpha\gamma_5 Q_j f^{abc}G^b_{\mu\nu}\tilde{G}^{c,\alpha}{}_\rho(\tilde{G}^b_{\mu\nu}\tilde{G}^{c,\alpha}{}_\rho)$$

QCD sum rules

Two-point correlation function

Definition: $\Pi(q^2) = i\int d^4x\, e^{iq\cdot x}\langle 0|\mathrm{T}[J(x)J^\dagger(0)]|0\rangle$

Hadron side: $\Pi^{\mathrm{Phe}}(q^2) = \int ds \frac{\rho(s)}{s-q^2-i\epsilon}$, where $\rho(s)$ is the spectral density.

One-pole narrow resonance approximation: $\rho(s) = f_X^2\delta(s-m_X^2) + \theta(s-s_0)\rho^{cont}(s)$

Operator product expansion (OPE) side:

$\Pi^{\mathrm{OPE}}(q^2) = \sum_n C_n(q^2)O_n$, where C_n's are the Wilson coefficients, O_n's are the condensates.

Quark-hadron duality: $\Pi^{\mathrm{Phe}}(q^2) = \Pi^{\mathrm{OPE}}(q^2)$

After Borel transform on both sides, we can extract the mass of the lowest resonance state

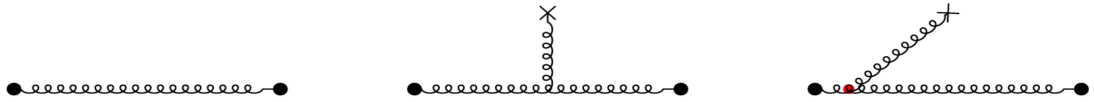
$$\mathcal{L}_k(s_0, M_B^2) = f_X^2(m_X^2)^k e^{-m_X^2/M_B^2} = \int_<^{s_0} ds\; e^{-\frac{s}{M_B^2}}\rho^{OPE}(s)s^k \;\Rightarrow\; m_X(s_0, M_B^2) = \sqrt{\frac{\mathcal{L}_1(s_0, M_B^2)}{\mathcal{L}_0(s_0, M_B^2)}}$$

QCD sum rules

Propagator for the heavy quark

$$iS_{ab}(p) = \frac{i\delta_{ab}}{\not{p} - m_Q} - \frac{i}{4} g_s T^r_{ab} G^r_{\mu\nu} \frac{\sigma^{\mu\nu}(\not{p} + m_Q) + (\not{p} + m_Q)\sigma^{\mu\nu}}{(p^2 - m_Q^2)^2},$$

Propagator for the gluon field strength $\boldsymbol{G^r_{\mu\nu} = \partial_\mu A^r_\nu - \partial_\nu A^r_\mu + g_s f^{rst} A^s_\mu A^t_\nu}$

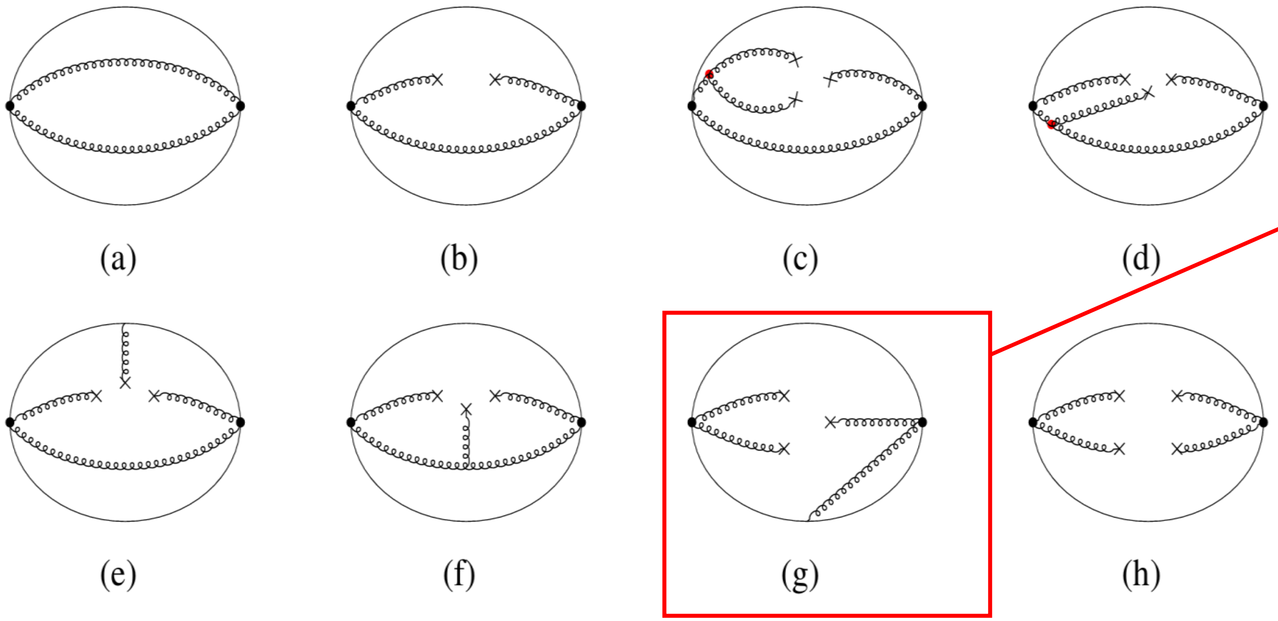


(a) (b) (c)

$$iS^{rs}_{\alpha\mu,\beta\nu}(p) = \frac{-i\delta^{rs}}{p^2 + i\epsilon}\left[g_{\mu\nu}p_\alpha p_\beta - g_{\mu\beta}p_\alpha p_\nu + (\mu \leftrightarrow \alpha,\ \nu \leftrightarrow \beta)\right]$$
$$+ \frac{i g_s f^{rst}}{(p^2 + i\epsilon)^2} G^{t,\rho\sigma}(0)\Big[2p_\alpha g_{\mu\rho}(p_\beta g_{\nu\sigma} - p_\nu g_{\beta\sigma}) + p_\alpha p_\sigma(g_{\mu\beta}g_{\nu\rho} - g_{\mu\nu}g_{\beta\rho})$$
$$+ (\mu \leftrightarrow \alpha,\ \nu \leftrightarrow \beta)\Big]$$
$$+ \frac{i g_s f^{rst}}{2(p^2 + i\epsilon)^2} G^{t,\rho\sigma}(0)\Big[2g_{\alpha\sigma}p_\rho(g_{\mu\beta}p_\nu - g_{\mu\nu}p_\beta) - p^2 g_{\alpha\sigma}(g_{\mu\beta}g_{\nu\rho} - g_{\mu\nu}g_{\beta\rho})$$
$$+ (\mu \leftrightarrow \alpha,\ \nu \leftrightarrow \beta)\Big],$$

QCD sum rules

Feynman diagrams for OPE



This diagram was neglected in the previous studies. There exists non-local divergence.

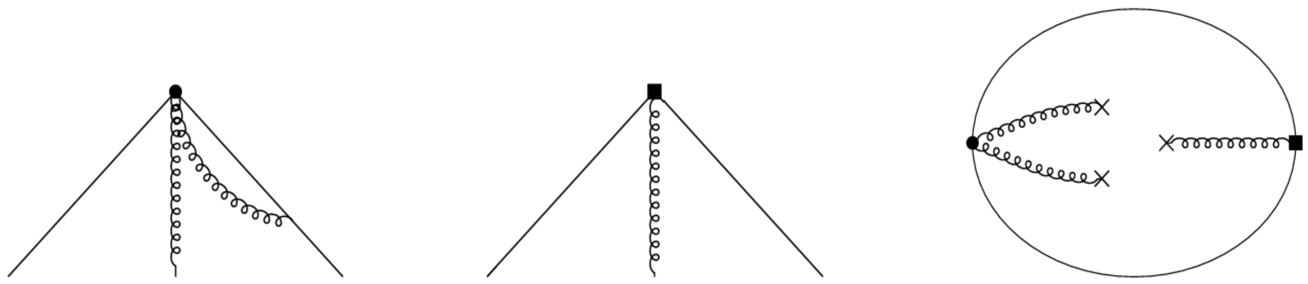
This kind of divergence cannot be canceled out by Borel transform.

Figure 2. The LO Feynman diagrams for the $\bar{Q}GGQ$ hybrid meson systems up to dimension eight gluon condensate. Diagrams related by symmetry are not shown.

QCD sum rules

Diagrammatic renormalization approach

Reference: *Renormalization* by Collins, *Foundations of quantum chromodynamics* by Muta and *Phys. Rev. D 106, 114023 (2022)*



(a) (b) (c)

Figure 3. Diagram (a) is a subgraph of figure 2(g), raising the non-local divergence exhibited in figure 2(g). Diagram (b) with the black square represents the counter term vertex generated from the divergence of diagram (a). Diagram (c) with the counter term vertex will eliminate the non-local divergence in figure 2(g).

Numerical analysis

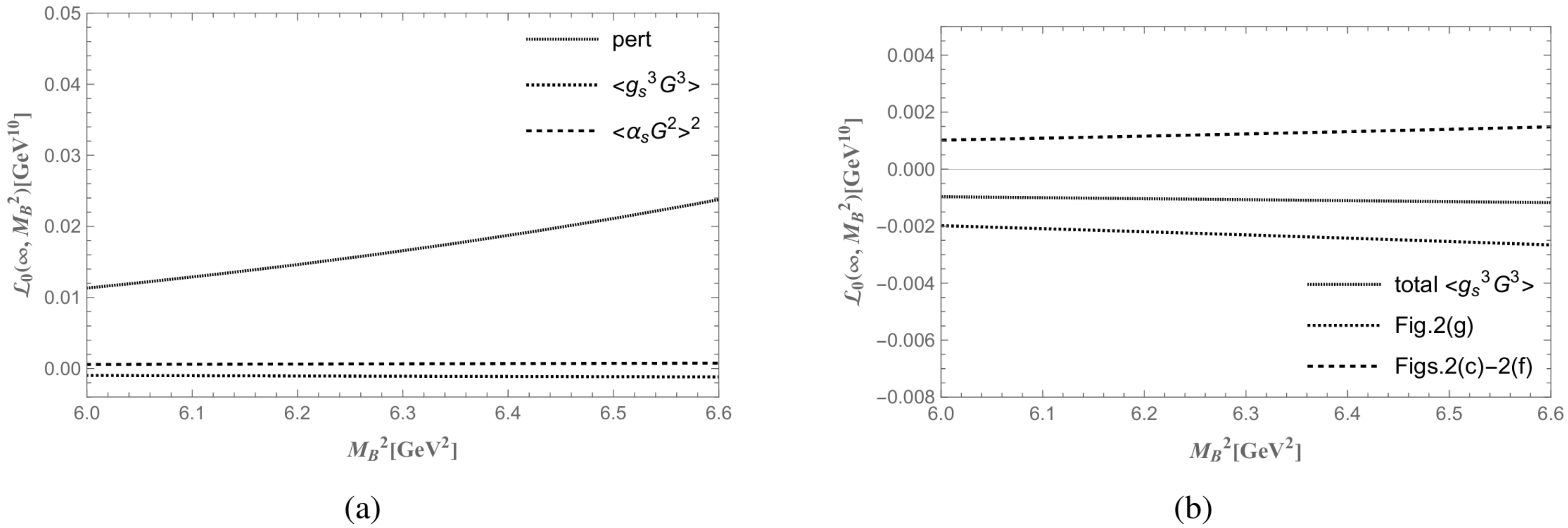


Figure 4. For the charmonium hybrid system with the interpolating current J^1_μ carrying $J^{PC} = 1^{-+}$: (a) convergence of the OPE series; (b) important contribution from the Feynman diagram figure 2(g) to the tri-gluon condensate.

Numerical analysis

Criterion:

$$R_{D=6} = \left| \frac{\mathcal{L}_0^{D=6}(\infty, M_B^2)}{\mathcal{L}_0^{total}(\infty, M_B^2)} \right| < 20\% \,,$$

$$R_{D=8} = \left| \frac{\mathcal{L}_0^{D=8}(\infty, M_B^2)}{\mathcal{L}_0^{total}(\infty, M_B^2)} \right| < 10\% \,,$$

$$\text{PC} = \left| \frac{\mathcal{L}_0^{total}(s_0, M_B^2)}{\mathcal{L}_0^{total}(\infty, M_B^2)} \right| > 40\% \,.$$

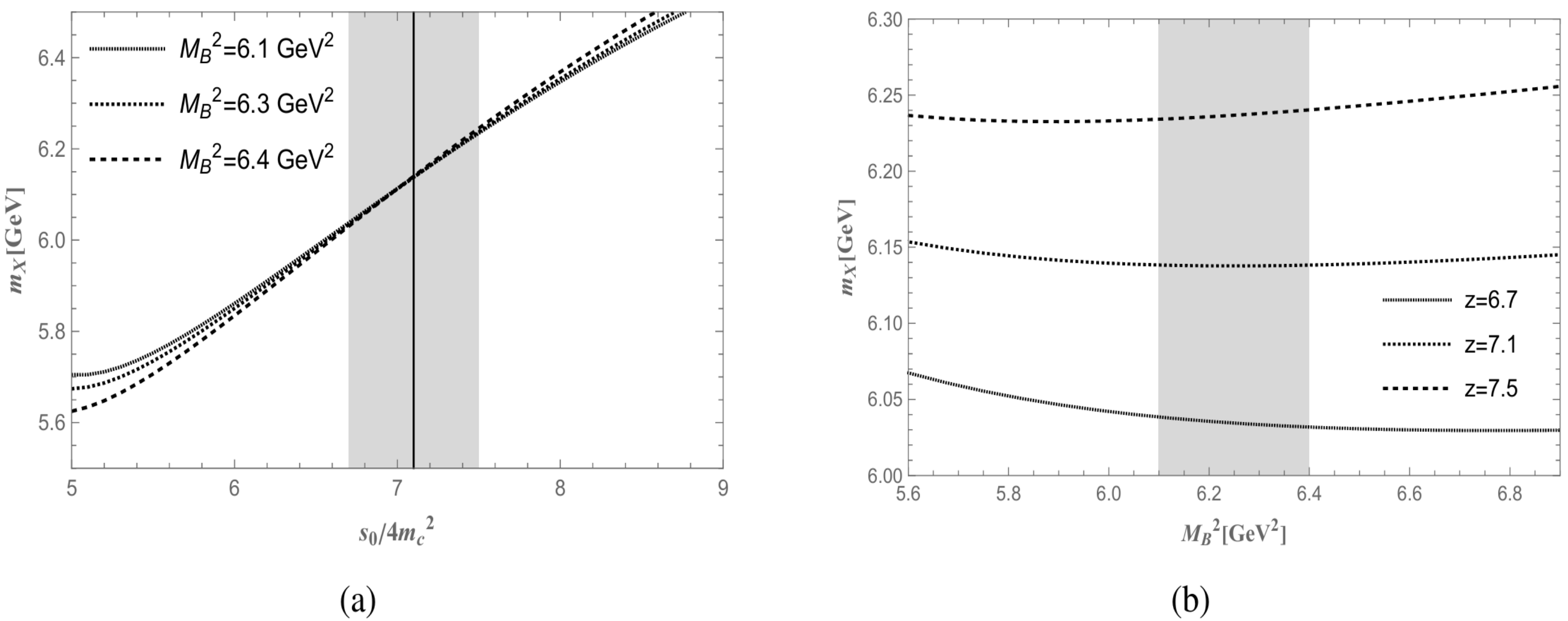


Figure 5. The variations of charmonium hybrid mass m_X with respect to $z = s_0/4m_c^2$ and M_B^2 for J_μ^1 with $J^{PC} = 1^{-+}$.

Results

Charmonium system

Currents	J^{PC}	Glueball operator	$s_0/(4m_c^2)$	$M_B^2(\mathrm{GeV}^2)$	$m_X(\mathrm{GeV})$	PC
$J_\mu^{1/2}$	1^{-+}	A	7.1 ± 0.4	$6.12 \sim 6.42$	6.14 ± 0.19	45.9%
$J_\mu^{4/5/6}$	1^{-+}	S	9.7 ± 0.52	$8.15 \sim 8.56$	7.21 ± 0.14	46.8%
$J_{\mu\nu\rho}^1$	2^{+-}	S	3.24 ± 0.53	$7.75 \sim 8.75$	4.22 ± 0.28	2.2%
$J_{\mu\nu\rho}^2$	2^{+-}	S	7.71 ± 0.44	$6.77 \sim 7.25$	6.41 ± 0.17	46.6%
$J_{\mu\nu\rho}^3$	2^{+-}	A	3.74 ± 0.28	$5.0 \sim 5.5$	4.55 ± 0.17	12%
$J_{\mu\nu\rho}^4$	2^{+-}	A	7.4 ± 0.37	$6.16 \sim 6.51$	6.33 ± 0.17	44.3%

A and S represent the antisymmetric and symmetric glueball operators in currents.

Masses for 1^{-+} are around 6.1 GeV and 7.2 GeV.

Masses for 2^{+-} are around 6.3 GeV and 6.4 GeV.

Results

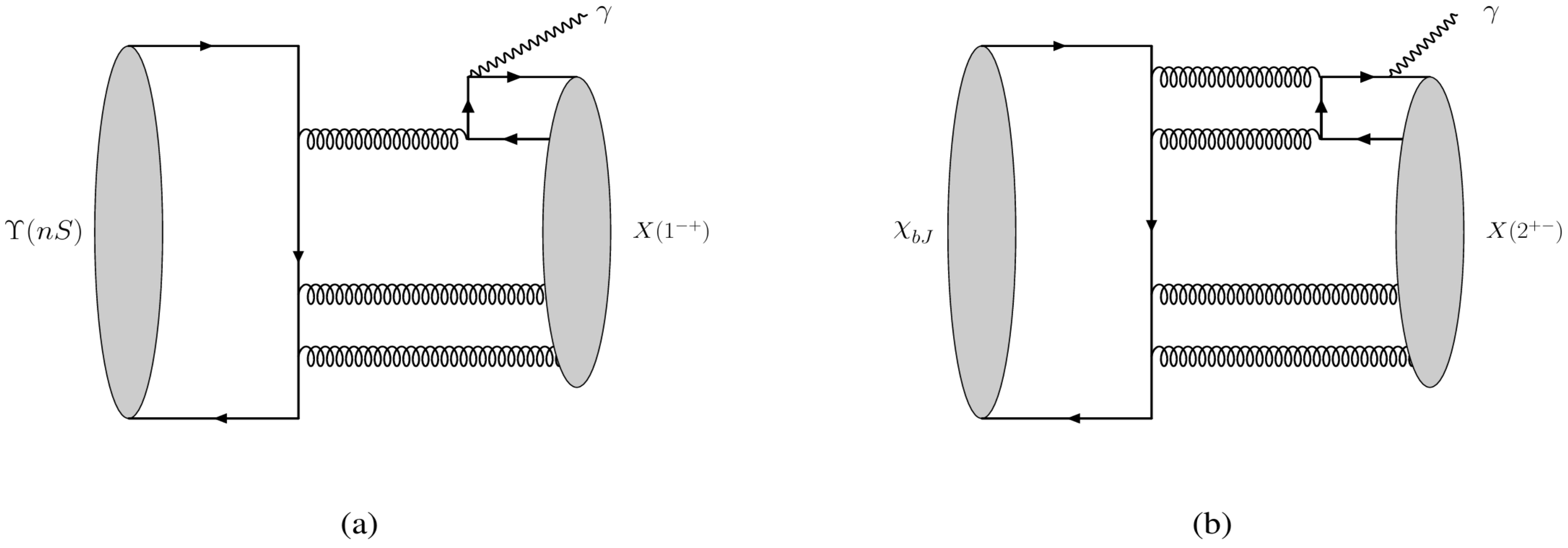
Bottomonium system

Currents	J^{PC}	Glueball operator	$s_0/(4m_b^2)$	$M_B^2(\mathrm{GeV}^2)$	m_X(GeV)	PC
$J_\mu^{1/2}$	1^{-+}	A	3.38 ± 0.09	$23.55 \sim 24.43$	14.26 ± 0.19	45.8%
$J_\mu^{4/5/6}$	1^{-+}	S	3.04 ± 0.12	$18.13 \sim 19.0$	13.71 ± 0.2	44.1%
$J_{\mu\nu\rho}^1$	2^{+-}	S	2.16 ± 0.08	$15.31 \sim 15.92$	11.67 ± 0.26	19%
$J_{\mu\nu\rho}^2$	2^{+-}	S	2.58 ± 0.11	$14.76 \sim 15.44$	12.58 ± 0.16	49.8%
$J_{\mu\nu\rho}^3$	2^{+-}	A	1.52 ± 0.16	$12.85 \sim 14.65$	9.85 ± 0.43	5.3%
$J_{\mu\nu\rho}^4$	2^{+-}	A	2.88 ± 0.13	$18.2 \sim 19.1$	13.31 ± 0.19	43.3%

Masses for 1^{-+} are around 13.7 GeV and 14.3 GeV.

Masses for 2^{+-} are around 12.6 GeV and 13.3 GeV.

Results



Possible production mechanisms for the charmonium systems with $J^{PC} = 1^{-+}$ and 2^{+-}.

With copious bottomonium mesons in Belle-II experiment, it is possible to detect the charmonium systems via the following radiative processes,

$$\Upsilon(nS) \to \gamma X(1^{-+}), \;\; \chi_{bJ} \to \gamma X(2^{+-})$$

Conclusion

- We studied the current operators for the double-gluon hybrid meson and constructed currents for the heavy double-gluon hybrid mesons with exotic quantum numbers $J^{PC} = 1^{-+}$ and 2^{+-}.
- The QCD sum rule method is applied to these currents to obtain the masses of the corresponding states. One additional tri-gluon condensate diagram needs to be renormalized.
- Belle-II: $\Upsilon(nS) \to \gamma X(1^{-+})$, $\chi_{bJ} \to \gamma X(2^{+-})$.
- Further investigations on the double-gluon hybrid mesons in various theoretical and phenomenological methods are expected.

Thank you!

Backup

We have

$$\tilde{G}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{a,\alpha\beta} \qquad G^a_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\tilde{G}^{a,\alpha\beta}$$

So

$$\tilde{G}^b_{\mu\nu}G^{c,\nu}{}_{\rho} = -G^b_{\rho\nu}\tilde{G}^{\nu,c}{}_{\mu} - \frac{1}{2}g_{\mu\rho}G^b_{\alpha\beta}\tilde{G}^{c,\alpha\beta} \qquad \tilde{G}^b_{\mu\nu}\tilde{G}^{c,\nu}{}_{\rho} = G^b_{\rho\nu}G^{\nu,c}{}_{\mu} + \frac{1}{2}g_{\mu\rho}G^b_{\alpha\beta}G^{c,\alpha\beta}$$

Then

$$d^{abc}\tilde{G}^b_{\mu\nu}G^{c,\nu}_{\rho} = d^{abc}G^b_{\rho\nu}\tilde{G}^{\nu,c}{}_{\mu} \Rightarrow d^{abc}\tilde{G}^b_{\mu\nu}G^{c,\nu}_{\rho} = -\frac{1}{4}g_{\mu\rho}d^{abc}G^b_{\alpha\beta}\tilde{G}^{c,\alpha\beta} = d^{abc}G^b_{\rho\nu}\tilde{G}^{\nu,c}{}_{\mu}$$

$$f^{abc}\tilde{G}^b_{\mu\nu}\tilde{G}^{c,\nu}_{\rho} = f^{abc}G^b_{\rho\nu}G^{\nu,c}{}_{\mu} = -f^{abc}G^b_{\mu\nu}G^c_{\nu\rho}$$

Thus, we have

$$J^1_\mu = -J^2_\mu \qquad J^5_\mu = j^6_\mu = -\frac{1}{4}J^4_\mu \qquad J^3_\mu = 0$$

Backup

Coupling relations

Operator with one Lorentz index

$$\left\langle 0\left|J_\mu\right|1^P(p)\right\rangle = Z_1^1\epsilon_\mu\,,$$
$$\left\langle 0\left|J_\mu\right|0^{(-P)}(p)\right\rangle = Z_1^0 p_\mu\,,$$

Operator with two Lorentz indices

$$\left\langle 0\left|J_{\mu\nu}\right|2^P(p)\right\rangle = Z_1^2\epsilon_{\mu\nu}\,,$$
$$\left\langle 0\left|J_{\mu\nu}\right|1^P(p)\right\rangle = Z_1^1\varepsilon_{\mu\nu\alpha\beta}\epsilon^\alpha p^\beta\,,$$
$$\left\langle 0\left|J_{\mu\nu}\right|1^{(-P)}(p)\right\rangle = Z_2^1\epsilon_\mu p_\nu + Z_3^1\epsilon_\nu p_\mu\,,$$
$$\left\langle 0\left|J_{\mu\nu}\right|0^P(p)\right\rangle = Z_1^0 g_{\mu\nu} + Z_2^0 p_\mu p_\nu\,,$$

Backup

Coupling relations

Operator with three Lorentz indices

$$\left\langle 0 \left| J_{\mu\nu\rho} \right| 3^P(p) \right\rangle = Z_1^3 \epsilon_{\mu\nu\rho} ,$$

$$\left\langle 0 \left| J_{\mu\nu\rho} \right| 2^{(-P)}(p) \right\rangle = Z_1^2 \epsilon_{\mu\nu} p_\rho + Z_2^2 \epsilon_{\mu\rho} p_\nu + Z_3^2 \epsilon_{\nu\rho} p_\mu ,$$

$$\left\langle 0 \left| J_{\mu\nu\rho} \right| 2^P(p) \right\rangle = Z_4^2 \varepsilon_{\mu\nu\tau\theta} \epsilon_\rho{}^\tau p^\theta + Z_5^2 \varepsilon_{\mu\rho\tau\theta} \epsilon_\nu{}^\tau p^\theta ,$$

$$\left\langle 0 \left| J_{\mu\nu\rho} \right| 1^P(p) \right\rangle = Z_1^1 \epsilon_\mu g_{\nu\rho} + Z_2^1 \epsilon_\nu g_{\mu\rho} + Z_3^1 \epsilon_\rho g_{\mu\nu} + Z_4^1 \epsilon_\mu p_\nu p_\rho + Z_5^1 \epsilon_\nu p_\mu p_\rho + Z_6^1 \epsilon_\rho p_\mu p_\nu ,$$

$$\left\langle 0 \left| J_{\mu\nu\rho} \right| 1^{(-P)}(p) \right\rangle = Z_7^1 \varepsilon_{\mu\nu\rho\tau} \epsilon^\tau + Z_8^1 \varepsilon_{\mu\nu\tau\lambda} \epsilon^\tau p^\lambda p_\rho + Z_9^1 \varepsilon_{\mu\rho\tau\lambda} \epsilon^\tau p^\lambda p_\nu ,$$

$$\left\langle 0 \left| J_{\mu\nu\rho} \right| 0^{(-P)}(p) \right\rangle = Z_1^0 p_\mu g_{\nu\rho} + Z_2^0 p_\nu g_{\mu\rho} + Z_3^0 p_\rho g_{\mu\nu} + Z_4^0 p_\mu p_\nu p_\rho ,$$

$$\left\langle 0 \left| J_{\mu\nu\rho} \right| 0^P(p) \right\rangle = Z_5^0 \varepsilon_{\mu\nu\rho\tau} p^\tau ,$$