

A gauge-invariant measure for gauge fields on $\mathbb{C}\mathbb{P}^2$

ANTONINA MAJ

THE GRADUATE CENTER, CUNY

amaj@gradcenter.cuny.edu

QCD 24: 27th High Energy Physics International Conference in QCD

Montpellier, 8 - 12th July 2024

➤ *Gauge and Scalar Fields on $\mathbb{C}\mathbb{P}^2$: A Gauge-Invariant Analysis I. The effective action from chiral scalars*

D. Karabali, A. Maj and V.P. Nair

Phys. Rev. D **106**, 085012 (2022) [arXiv:2110.11926]

➤ *Gauge and Scalar Fields on $\mathbb{C}\mathbb{P}^2$: A Gauge-Invariant Analysis II. The measure for gauge fields and a 4d WZW theory*

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Motivation

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- Towards an effective theory for low energy 4d Yang-Mills theory

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- On \mathbb{CP}^2 gauge-invariant parametrization

$$\mathcal{Z}[A] = \int [dA] e^{-S[A]} = \int \int [dH] e^{-\tilde{S}[H]}$$

$A = A(H, U)$
↙

~~$[dU]$~~
↓
gauge modes

Outline

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1. Parametrization of the gauge fields

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3. Results and future directions

2d toy model

2d Abelian fields: $A_\mu(x^\mu)$, $\mu = 1, 2$

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2) Gauge dof: $\text{Re}(f)$, gauge-invariant dof: $\text{Im}(f)$

4d case

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2. Choose $\mathbb{C}\mathbb{P}^2$ as the 2d complex manifold

\mathbb{CP}^2 basics

Homogenous coordinates: $Z = (Z_1, Z_2, Z_3), \quad Z \neq 0$

$$(Z_1, Z_2, Z_3) \equiv (\lambda Z_1, \lambda Z_2, \lambda Z_3), \quad \lambda \in \mathbb{C} - \{0\}$$

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$$Z = Z_3 \left(\frac{Z_1}{Z_3}, \frac{Z_2}{Z_3}, 1 \right) \equiv Z_3(z^1, z^2, 1)$$

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Metric:
$$ds^2 = \frac{dz \cdot d\bar{z}}{(1 + z \cdot \bar{z})} - \frac{dz \cdot \bar{z}d\bar{z} \cdot z}{(1 + z \cdot \bar{z})^2} = g_{a\bar{a}}dz^a d\bar{z}^{\bar{a}}$$

Measure:
$$d\mu = \frac{2}{\pi^2} \frac{d^4x}{(1 + z \cdot \bar{z})^3}$$

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1. 4d Euclidean space has no complex structure
2. Choose $\mathbb{C}P^2$ as the 2d complex manifold
 - finite space with flat space limit
 - group theoretic structure as $SU(3)/U(2)$ facilitates a parametrization of the gauge fields

gauge fields on $\mathbb{C}\mathbb{P}^2$

Abelian gauge fields:

$$A_a = \nabla_a f(g) + g_{a\bar{a}} \epsilon^{\bar{a}b} \bar{\nabla}_{\bar{b}} \chi(g)$$
$$\bar{A}_{\bar{a}} = -\bar{\nabla}_{\bar{a}} \bar{f}(g) - g_{\bar{a}a} \epsilon^{ab} \nabla_b \bar{\chi}(g)$$

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Non-Abelian gauge fields:

$$A_a = -\nabla_a M M^{-1} + M^{\dagger-1} \left(g_{a\bar{a}} \epsilon^{\bar{a}\bar{b}} \bar{\nabla}_{\bar{b}} \chi \right) M^{\dagger}$$

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gauge group $SU(N)$: $M, M^{\dagger} \in SL(N, \mathbb{C})$

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- Under gauge transformations $U \in SU(N)$:

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pure gauge
parameters: $U \in SU(N)$

gauge-invariant
parameters: $\rho, \chi, \bar{\chi}$

$$H = M^{\dagger} M = \rho^2$$

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Measure for space of gauge potentials

Measure for gauge potentials: $dV = \prod_x [\det(g_{a\bar{a}}) dA_a^\alpha d\bar{A}_{\bar{a}}^\alpha]$

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$$\xi = (\theta, \theta^\dagger, \delta\tilde{\chi}, \delta\tilde{\chi}^\dagger) \equiv (\delta M M^{-1}, M^{\dagger-1} \delta M^\dagger, M^{\dagger-1} \delta\chi M^\dagger, M \delta\bar{\chi} M^{-1})$$

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metric: $ds^2 = \frac{1}{2} \int \xi^{\dagger\alpha} \mathcal{M}_{\alpha\beta} \xi^\beta$

Jacobian:

$$\mathcal{M} = \begin{pmatrix} -\bar{D} \cdot D & -2\epsilon^{ab} \bar{D}_{\bar{a}}(M^{\dagger-1} \chi M^{\dagger}) \bar{D}_{\bar{b}} & 0 & -\bar{D}(M^{\dagger-1} \chi M^{\dagger}) \cdot D \\ 2\epsilon^{ab} D_a(M \bar{\chi} M^{-1}) D_b & -D \cdot \bar{D} & D(M \bar{\chi} M^{-1}) \cdot \bar{D} & 0 \\ 0 & -D \cdot \bar{D}(M^{\dagger-1} \chi M^{\dagger}) & -D \cdot \bar{D} & 0 \\ \bar{D} \cdot D(M \bar{\chi} M^{-1}) & 0 & 0 & -\bar{D} \cdot D \end{pmatrix}$$

where $D_a \Phi = \nabla_a \Phi + [-\nabla_a M M^{-1}, \Phi]$, $\bar{D}_{\bar{a}} \Phi = \bar{\nabla}_{\bar{a}} \Phi + [M^{\dagger-1} \bar{\nabla}_{\bar{a}} M^{\dagger}, \Phi]$

Measure

Measure for the gauge orbit space: $dV[\mathcal{C}] = e^{\Gamma(H, \chi, \bar{\chi})} [d\mu(H) d\chi d\bar{\chi}]$

$$\begin{aligned} \Gamma(H, \chi, \bar{\chi}) &= \frac{7}{2\pi r^2} S_{\text{wzw}}(H) + \left(-\frac{1}{4\epsilon} + \frac{1}{2r^2} \log \epsilon \right) \int \text{Tr} (\bar{\nabla} \chi \cdot H \nabla \bar{\chi} H^{-1}) \\ &+ \frac{\log \epsilon}{12} \int \text{Tr} \left[(\bar{\nabla} (\nabla H H^{-1}))^2 + (\bar{\nabla} \chi \cdot H \nabla \bar{\chi} H^{-1})^2 \right. \\ &\quad \left. + g^{a\bar{a}} g^{b\bar{b}} \bar{\nabla}_{\bar{a}} (\nabla_b H H^{-1}) [H \nabla_a \bar{\chi} H^{-1}, \bar{\nabla}_{\bar{b}} \chi] \right. \\ &\quad \left. - g^{a\bar{a}} g^{b\bar{b}} \bar{\nabla}_{\bar{a}} (H \nabla_b \bar{\chi} H^{-1}) \mathcal{D}_a \bar{\nabla}_{\bar{b}} \chi \right] \\ &+ \text{finite terms} \end{aligned}$$

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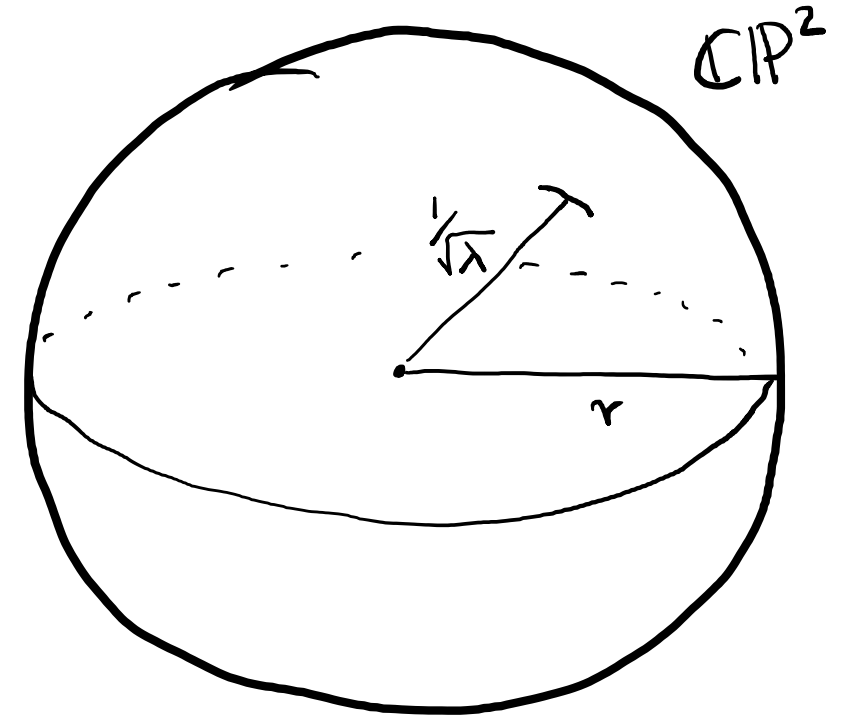
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 \end{aligned}$$

Low energy effective theory

1. Introduce an IR cutoff λ s.t. $1/\lambda < r^2$

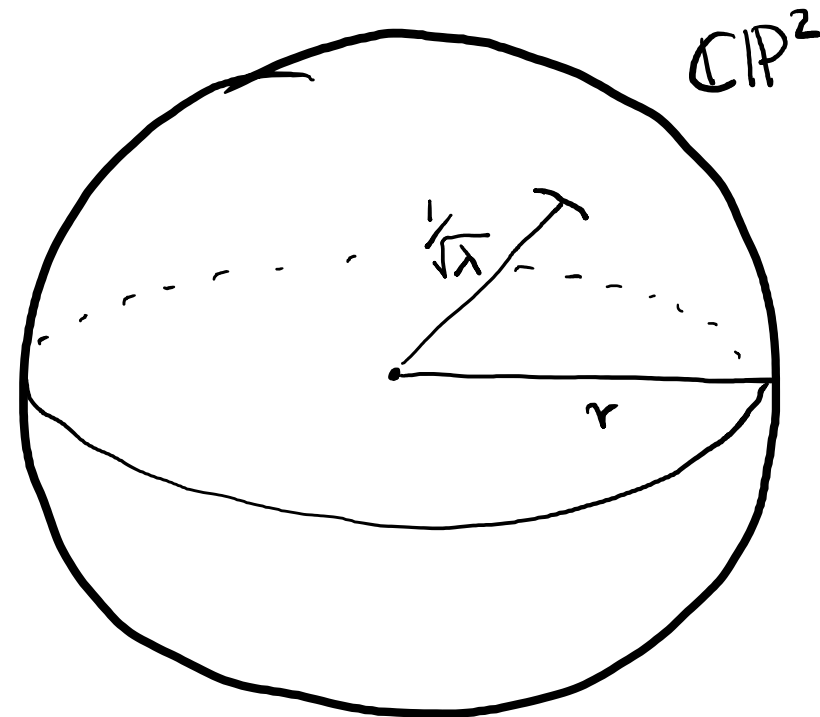
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2. Take the flat limit of \mathbb{CP}^2 $r \rightarrow \infty$

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 & \quad \left. + g^{a\bar{a}} g^{b\bar{b}} \bar{\nabla}_{\bar{a}} (\nabla_b H H^{-1}) [H \nabla_a \bar{\chi} H^{-1}, \bar{\nabla}_{\bar{b}} \chi] \right. \\
 & \quad \left. - g^{a\bar{a}} g^{b\bar{b}} \bar{\nabla}_{\bar{a}} (H \nabla_b \bar{\chi} H^{-1}) \mathcal{D}_a \bar{\nabla}_{\bar{b}} \chi \right] \\
 & + \text{finite terms } \mathcal{O}(k^2/\lambda)
 \end{aligned}$$

Low energy effective theory

Measure for the gauge orbit space: $dV[\mathcal{C}] = e^{\Gamma(H, \chi, \bar{\chi})} [d\mu(H) d\chi d\bar{\chi}]$

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$k^2 \ll \lambda$

Outline

1. Parametrization of the gauge fields
2. Measure of integration for the gauge orbit space
3. Results and future directions

Results

Measure for the gauge orbit space: $dV[\mathcal{C}] = e^{\Gamma(H, \chi, \bar{\chi})} [d\mu(H) d\chi d\bar{\chi}]$

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$$\Gamma_{\text{log}} \sim \log \epsilon \int (\partial A)^2$$

Mass terms

$$\Gamma(H, \chi, \bar{\chi}) = \frac{\lambda}{2\pi} S_{\text{wzw}}(H) - \frac{1}{4\epsilon} \int \text{Tr} (\bar{\nabla} \chi \cdot H \nabla \bar{\chi} H^{-1}) + \dots$$

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$$\rightarrow \frac{\lambda}{2\pi} S_{\text{wzw}}(H) + \mu_{\text{ren}} \int \text{Tr} (\bar{\nabla} \chi \cdot H \nabla \bar{\chi} H^{-1}) + \dots$$

Dimensional transmutation

$$\Gamma(H, \chi, \bar{\chi}) = \frac{\lambda}{2\pi} S_{\text{wzw}}(H) + \mu_{\text{ren}} \int \text{Tr} (\bar{\nabla} \chi \cdot H \nabla \bar{\chi} H^{-1})$$

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$$Z = \int [d\mu(H)][d\chi][d\bar{\chi}] e^{\Gamma(H, \chi, \bar{\chi}) - \frac{1}{2g^2} \int \text{Tr} F^2} \quad \rightarrow \quad \Lambda_{\text{QCD}} = \Lambda_{\text{QCD}}(\lambda, \mu_{\text{Ren}})$$

Gluon self-energy

$$\Gamma(H, \chi, \bar{\chi}) = \frac{\lambda}{2\pi} S_{\text{wzw}}(H) + \mu_{\text{ren}} \int \text{Tr} (\bar{\nabla} \chi \cdot H \nabla \bar{\chi} H^{-1})$$

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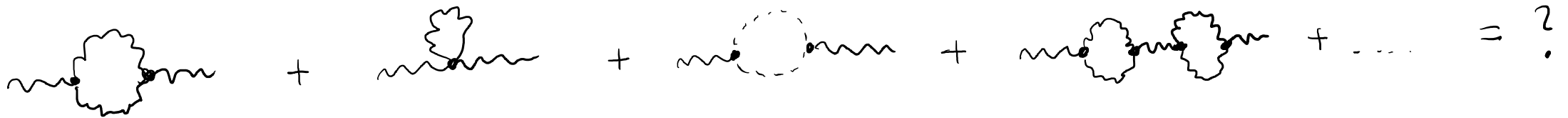
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Mass term in 4d Euclidean space:

$$\int \text{Tr} A_i \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) A_j + \dots \sim \int \text{Tr} F \left(\frac{1}{-D \cdot D} \right) F ?$$

Gluon self-energy

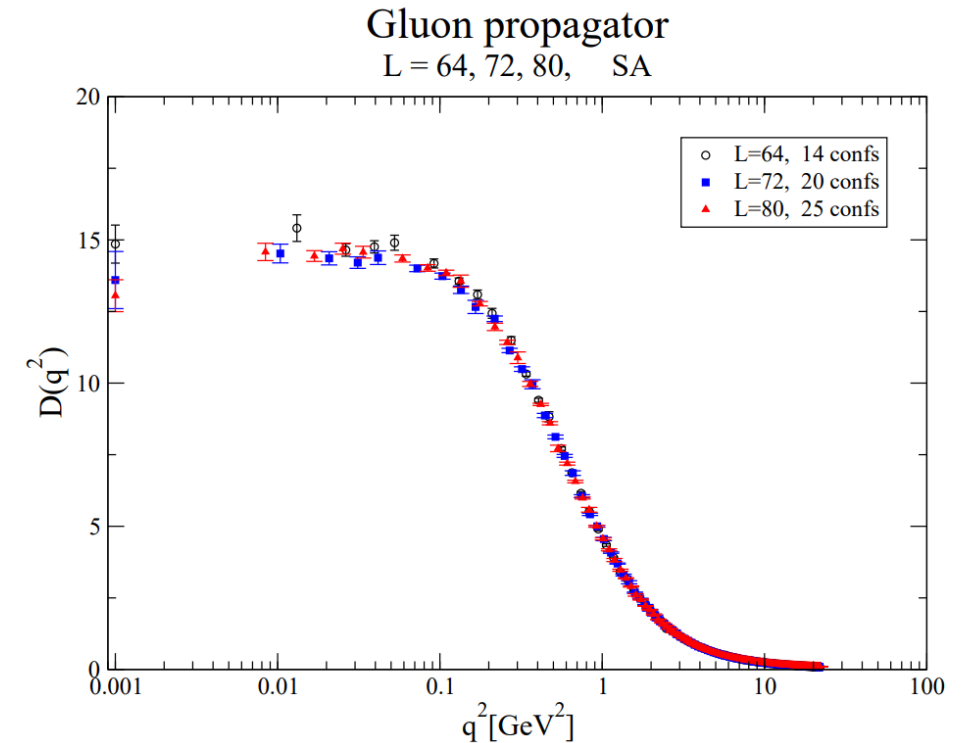
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Gluon self-energy

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➤ Lattice simulation evidence

- I.L. Bogolubsky, et.al., Phys.Lett. B676, 69-73 (2009)
- I.L. Bogolubsky et.al., PoSLAT2007:290 (2007)
- P.O. Bowman, et.al., Phys.Rev. D76:094505 (2007)
and more...



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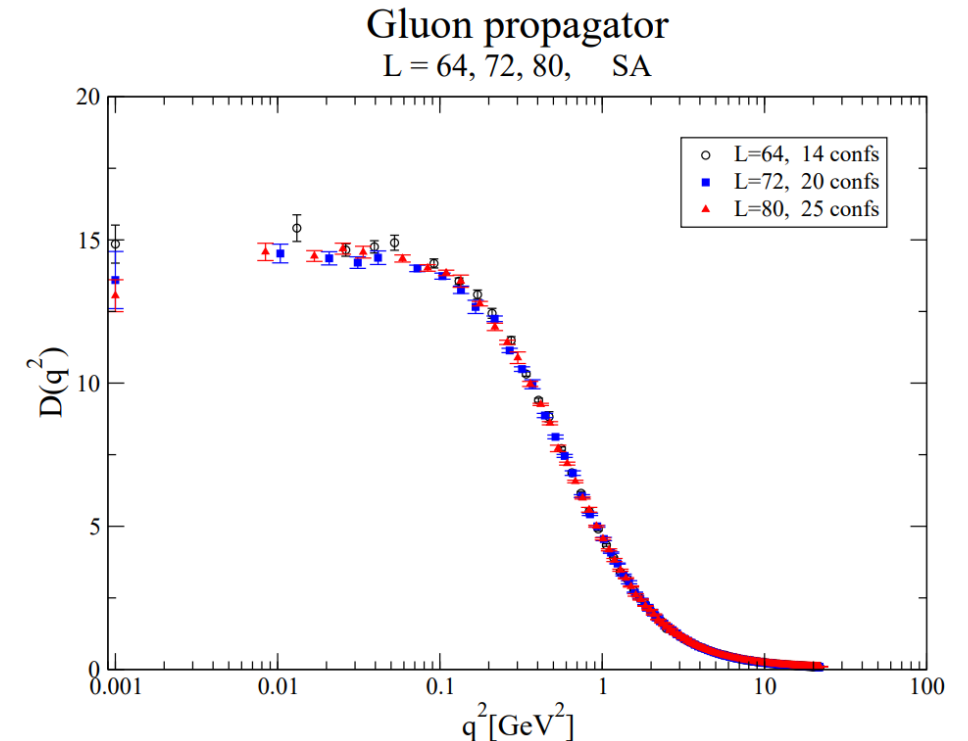
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➤ Schwinger-Dyson calculations of gluon self-energy

- A.C. Aguilar et.al., Front.Phys. 11(2), 111203 (2016)
- A.C. Aguilar et.al., Phys.Rev. D78:025010 (2008)



WZW term

$$\Gamma(H, \chi, \bar{\chi}) = \frac{\lambda}{2\pi} S_{\text{wzw}}(H) + \mu_{\text{ren}} \int \text{Tr} (\bar{\nabla} \chi \cdot H \nabla \bar{\chi} H^{-1}) + \dots$$

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$$Z \simeq \int [d\mu(H)] e^{\frac{\lambda}{2\pi} S_{\text{wzw}}(H) - \frac{1}{2g^2} \int \text{Tr} F^2(H)}$$

4d WZW action

$$S_{\text{wzw}}(H) = \frac{1}{2\pi} \int g^{a\bar{a}} \text{Tr}(\nabla_a H \bar{\nabla}_{\bar{a}} H^{-1}) - \frac{i}{24\pi} \int \omega \wedge \text{Tr}(H^{-1} dH)^3$$

Kähler 2-form on $\mathbb{C}\mathbb{P}^2$: $\omega = i g_{a\bar{a}} dz^a d\bar{z}^{\bar{a}}$

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- 4d-WZW theory first considered by Simon Donaldson (1980s).
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- Low energy dynamics dominated by anti-self-dual instantons.

Casimir energy

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Lattice simulations of Casimir energy for 4d Yang-Mills theory

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with $m^2 = \frac{\lambda g^2}{8\pi^2}$

WZW theory and confinement

WZW theory and confinement

In 2+1 Yang Mills theory can show area law of Wilson loop:

$$\begin{aligned}
 \langle W(C) \rangle &= \langle \text{Tr} \mathcal{P} e^{\oint_C \nabla H H^{-1}} \rangle \approx \text{Tr} \exp \left[\frac{1}{2} \langle \oint_C \nabla H H^{-1} \oint_C \nabla H H^{-1} \rangle \right] \\
 &= \text{Tr} \exp \left[\mathcal{N} \int d\mu(H) e^{2c_A S_{\text{wzw}}^{(2d)}(H) - \frac{8\pi}{e^4 c_A} \int \text{Tr}(\bar{\nabla}(\nabla H H^{-1}))^2} \left(\frac{1}{2} \oint_C \nabla H H^{-1} \oint_C \nabla H H^{-1} \right) \right] \\
 &\sim e^{-\sigma_R \text{Area}(C)}, \quad \sigma_R = e^4 \frac{c_A c_R}{4\pi}
 \end{aligned}$$

D. Karabali, C. Kim, V.P. Nair, *Phys. Lett.* **B434**, 103 (1998)

WZW theory and confinement

In 4d Yang Mills theory?

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 \end{aligned}$$

Calculations too complicated here...

Thank you!

$$\mathbb{C}\mathbb{P}^2 = SU(3)/U(2)$$

- functions on $SU(3)$ in terms of Wigner D-matrices:

$$F(g) = \sum_{J,A} C_A^{(J)} \mathcal{D}_{A,B}^{(J)}(g) = \sum_{J,A} C_A^{(J)} \langle J, A | \hat{g} | J, B \rangle$$

- $U(2)$ subgroup: local isotropy group for $\mathbb{C}\mathbb{P}^2$

➤ scalars, vectors, tensors are functions on $SU(3)$ that behave in the right way under the $U(2)$ subgroup

Scalar functions on $\mathbb{C}\mathbb{P}^2$: $f(g) = f(gh), \quad h \in U(2)$ subgroup

Vector fields on $\mathbb{C}\mathbb{P}^2$: $\nabla_a f(g), \quad \bar{\nabla}_{\bar{a}} f(g), \quad a = 1, 2$

→ $SU(2)$ doublets with $Y = \pm 1$