

QCD bounds on $a_\mu^{\text{HVP,LO}}$

arXiv: 2404.08591 [hep-ph]

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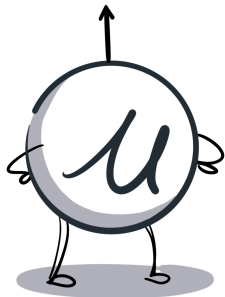
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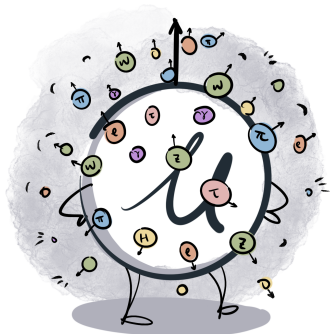
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Muon Anomalous Magnetic Moment a_μ



$$g = 2$$



$$g \neq 2$$

SL.

Illustration by Siyuan Li (inspired by "The Muon $g-2$ Anomaly Explained" from [aps.org](https://www.aps.org))

$$a_\mu = (g - 2)_\mu / 2$$

- Fermilab Muon $g - 2$ Experiment: error reduced by a factor of 2, disagree with theory (Aoyama, et al. 2020) at 5.0σ
- CMD-3 detector for pion

...

US:

QCD bounds on leading-order hadronic vacuum polarization contributions to the muon anomalous magnetic moment

Siyuan Li, T.G. Steele, J. Ho, R. Raza, K. Williams, R.T. Kleiv

2404.08591 [hep-ph]

[Revision Accepted]

Hadronic Vacuum Polarization (HVP) contributions to a_μ

$$a_\mu^{\text{SM}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{had}}$$

The HVP term arises from the interactions between the muon and the hadronic content (quark and gluon) of the vacuum via the strong force.

⇒ Quantum Chromodynamics Sum Rules (QCDSR)

Methodology

Finite-energy QCD sum rule (FESR)

LO dispersion integral for the contributions to a_μ from HVP

$$a_\mu^{\text{HVP,LO}} = \frac{1}{4\pi^3} \int_{4m_\pi^2}^{\infty} \sigma^H(t) K(t) dt \approx \frac{4m_\mu^2 \alpha^2}{3\pi} \int_{4m_\pi^2}^{\infty} \frac{1}{t^2} \text{Im}\Pi^H(t) dt$$

Finite-energy QCD sum rule (FESR) structure

$$F_k(s_0) = \int_{t_0}^{s_0} \frac{1}{\pi} \text{Im}\Pi^H(t) t^k dt,$$

$$\Rightarrow a_\mu^{\text{QCD}} \approx \frac{4m_\mu^2 \alpha^2}{3} F_{-2}(\infty)$$

Hölder inequality (Lower Bound)

The general Hölder inequality

$$\left| \int_{t_1}^{t_2} f(t) g(t) d\mu \right| \leq \left(\int_{t_1}^{t_2} |f(t)|^p d\mu \right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |g(t)|^q d\mu \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

With $d\mu = \frac{1}{\pi} \text{Im} \Pi^H(t) dt$ and careful choice of functions $f(t)$, $g(t)$:

$$F_{\alpha+\beta}(s_0) \leq [F_{\alpha p}(s_0)]^{\frac{1}{p}} \left[F_{\frac{\beta p}{p-1}}(s_0) \right]^{\frac{p-1}{p}}.$$

The most restrictive lower bound (from the Cauchy-Schwarz inequality) :

$$F_{-2} \geq \frac{F_0^3}{F_1^2}.$$

Upper Bound

Method inspired by *F. Dalfovo and S. Stringari, Phys. Rev. B 46, 3991 (1992)*.

$$\int_{t_0}^{s_0} \frac{1}{t^2} [1 + At]^2 \text{Im}\Pi^H(t) dt \leq \begin{cases} \frac{1}{t_0} \int_{t_0}^{s_0} \frac{1}{t} [1 + At]^2 \text{Im}\Pi^H(t) dt \\ \frac{1}{t_0^2} \int_{t_0}^{s_0} [1 + At]^2 \text{Im}\Pi^H(t) dt \end{cases}$$

$$\text{eg. } \int_{t_0}^{s_0} \left[\frac{1}{t^2} + \frac{2A}{t} + A^2 \right] \text{Im}\Pi^H(t) dt \sim F_{-2} + 2AF_{-1} + A^2 F_0$$

By extremizing A to obtain the relations:

$$a_\mu^{\text{QCD}} \leq \frac{4m_\mu^2 \alpha^2}{3} \begin{cases} F_{-1}^{(B)}/t_0 - \frac{(F_0/t_0 - F_{-1}^{(B)})^2}{F_1/t_0 - F_0} \\ F_0/t_0^2 - \frac{(F_1/t_0^2 - F_{-1}^{(B)})^2}{F_2/t_0^2 - F_0} \end{cases}, \quad F_{-1}^{(B)} \equiv \frac{F_0}{t_0} - \frac{(F_1/t_0 - F_0)^2}{(F_2/t_0 - F_1)}.$$

Our Constraints

Constraint: the Cauchy-Schwarz inequality (*i.e.*, the Hölder inequality with $p = 2$, $\alpha = \frac{k+1}{2}$ and $\beta = \frac{k-1}{2}$)

$$F_k^2 \leq F_{k+1} F_{k-1}.$$

$$\xi \frac{4m_\mu^2 \alpha^2}{3} \frac{F_0^3(s_0)}{F_1^2(s_0)} \leq a_\mu^{\text{QCD}} \leq \frac{4m_\mu^2 \alpha^2}{3} \left\{ \begin{array}{l} F_{-1}^{(B)}/t_0 - \frac{(F_0/t_0 - F_{-1}^{(B)})^2}{F_1/t_0 - F_0} \\ F_0/t_0^2 - \frac{(F_1/t_0^2 - F_{-1}^{(B)})^2}{F_2/t_0^2 - F_0} \end{array} \right. ,$$

where $\xi = 0.83$.

The QCD correlation function for the (single) light quark vector current $j_\mu(x) = \bar{q}(x)\gamma_\mu q(x)$:

$$\begin{aligned}\Pi(Q^2) &= \frac{1}{4\pi^2} \Pi^{\text{pert}}(Q^2) - \frac{3m_q^2(\nu)}{2\pi^2 Q^2} + 2\langle m_q \bar{q}q \rangle \frac{1}{Q^4} \left(1 + \frac{1}{3} \frac{\alpha_s(\nu)}{\pi} \right) \\ &\quad + \frac{1}{12\pi} \langle \alpha_s G^2 \rangle \frac{1}{Q^4} \left(1 + \frac{7}{6} \frac{\alpha_s(\nu)}{\pi} \right) - \frac{224}{81} \pi \alpha_s \langle \bar{q}q \rangle \frac{1}{Q^6}.\end{aligned}$$

The perturbative contributions:

$$\frac{1}{\pi} \text{Im} \Pi^{\text{pert}}(t, \nu) = 1 + \sum_{n=1}^{\infty} \frac{\alpha_s(\nu)^n}{\pi} \sum_{m=0}^{n-1} T_{n,m} \log^m \left(\frac{\nu^2}{t} \right).$$

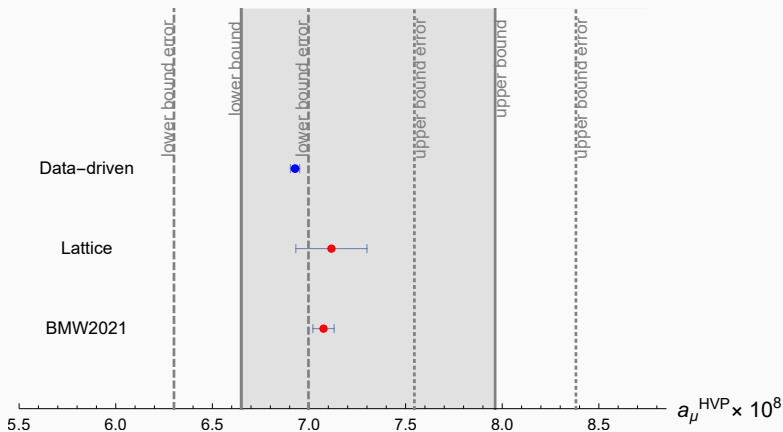
QCD Inputs (FESR)

$$F_0(s_0) = \frac{1}{4\pi^2} \left[1 + \frac{\alpha_s(\nu)}{\pi} T_{1,0} + \left(\frac{\alpha_s(\nu)}{\pi} \right)^2 (T_{2,0} + T_{2,1}) + \left(\frac{\alpha_s(\nu)}{\pi} \right)^3 (T_{3,0} + T_{3,1} + 2T_{3,2}) \right. \\ \left. + \left(\frac{\alpha_s(\nu)}{\pi} \right)^4 (T_{4,0} + T_{4,1} + 2T_{4,2} + 6T_{4,3}) \right] s_0 - \frac{3}{2\pi^2} m_q(\nu)^2,$$

$$F_1(s_0) = \frac{1}{8\pi^2} \left[1 + \frac{\alpha_s(\nu)}{\pi} T_{1,0} + \left(\frac{\alpha_s(\nu)}{\pi} \right)^2 \left(T_{2,0} + \frac{1}{2} T_{2,1} \right) + \left(\frac{\alpha_s(\nu)}{\pi} \right)^3 \left(T_{3,0} + \frac{1}{2} T_{3,1} + \frac{1}{2} T_{3,2} \right) \right. \\ \left. + \left(\frac{\alpha_s(\nu)}{\pi} \right)^4 \left(T_{4,0} + \frac{1}{2} T_{4,1} + \frac{1}{2} T_{4,2} + \frac{3}{4} T_{4,3} \right) \right] s_0^2 \\ - 2\langle m_q \bar{q}q \rangle \left(1 + \frac{1}{3} \frac{\alpha_s(\nu)}{\pi} \right) - \frac{1}{12\pi} \langle \alpha_s G^2 \rangle \left(1 + \frac{7}{6} \frac{\alpha_s(\nu)}{\pi} \right),$$

$$F_2(s_0) = \frac{1}{12\pi^2} \left[1 + \frac{\alpha_s(\nu)}{\pi} T_{1,0} + \left(\frac{\alpha_s(\nu)}{\pi} \right)^2 \left(T_{2,0} + \frac{1}{3} T_{2,1} \right) + \left(\frac{\alpha_s(\nu)}{\pi} \right)^3 \left(T_{3,0} + \frac{1}{3} T_{3,1} + \frac{2}{9} T_{3,2} \right) \right. \\ \left. + \left(\frac{\alpha_s(\nu)}{\pi} \right)^4 \left(T_{4,0} + \frac{1}{3} T_{4,1} + \frac{2}{9} T_{4,2} + \frac{2}{9} T_{4,3} \right) \right] s_0^3 - \frac{224}{81} \pi \alpha_s \langle \bar{q}q \rangle.$$

Results



*supplemented with charmonium and bottomonium resonance contributions ¹ for comparison.

¹A. Keshavarzi, D. Nomura, and T. Teubner, Phys. Rev. D 101, 014029 (2020).
T. Aoyama et al., Physics Reports 887, 1 (2020).
S. Borsanyi et al., Nature 593, 51 (2021), [arXiv:2002.12347 [hep-lat]].

- Construct constraints on $a_\mu^{\text{HVP,LO}}$ using FESR and Hölder inequalities

$$(657.0 \pm 34.8) \times 10^{-10} \leq a_\mu^{\text{HVP,LO}} \leq (788.4 \pm 41.8) \times 10^{-10};$$

- *Laplace Sum Rule approach;
- Optimization methodology for s_0^{opt} : flavour-separated approach;
- Bridge the gap between LQCD and data-driven approaches;
- A possible resolution of the tension between LQCD and data-driven determinations of $a_\mu^{\text{HVP,LO}}$.

**QCD bounds on leading-order hadronic vacuum polarization
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Thank you!

Questions?

Backup slides: QCD Sum Rules (QCDSR)

- Two-point correlation function $\Pi(q^2)$
- Operator Product Expansion (OPE)

$$\Pi(q^2) \sim \sum_n C_n(q^2) \langle \mathcal{O}_n \rangle = C_1(q^2) \mathbf{1} + C_4(q^2) \langle G^2 \rangle + \dots,$$

- Dispersion Relation

$$\begin{aligned} \Pi(Q^2) = & \Pi(0) + Q^2 \Pi'(0) + \frac{1}{2} Q^4 \Pi''(0) + \dots + \frac{1}{n} Q^{2n} \Pi^{(n)}(0) \\ & + Q^{2(n+1)} \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{\rho^{\text{had}}(s)}{s^{n+1}(s+Q^2)}, \end{aligned}$$

- **Extract hadronic parameters** by equating the correlator OPE (LHS) and the phenomenological representation (RHS)

Regularize divergence:

- 4-dimensional integral $\rightarrow d$ -dimensional
- Expand in $d = 4 \pm 2\epsilon$ dimensions as $\epsilon \rightarrow 0$
- divergences are in $\mathcal{O}(\frac{1}{\epsilon})$ terms

Backup slides: Laplace Sum Rule Approach

QCD Laplace sum-rules

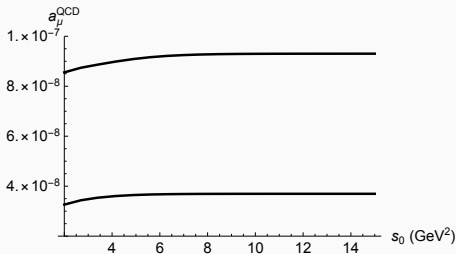
$$L_k(\tau, s_0) = \int_{t_0}^{s_0} \frac{1}{\pi} \text{Im}\Pi^H(t) t^k e^{-t\tau} dt.$$

Approximate Kernel function near $t = t'$:

$$K(t) \approx \mathcal{K}(t, t') = K(t') e^\zeta \left[a_1 \left(\frac{t}{t'} \right) + a_2 \left(\frac{t}{t'} \right)^2 + a_3 \left(\frac{t}{t'} \right)^3 \right] e^{-\zeta t/t'},$$

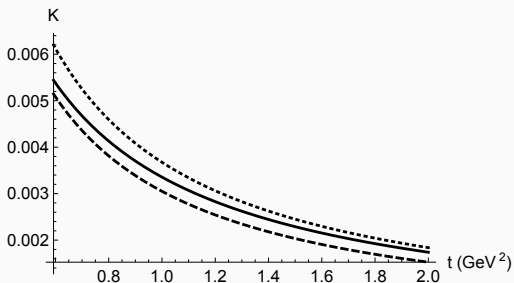
$$a_\mu^{\text{QCD}} \approx 4\alpha^2 K(\zeta/\tau) \frac{\tau}{\zeta} e^\zeta \left[a_1 L_0(\tau, s_0) + a_2 \frac{\tau}{\zeta} L_1(\tau, s_0) + a_3 \left(\frac{\tau}{\zeta} \right)^2 L_2(\tau, s_0) \right],$$

where $a_1 + a_2 + a_3 = 1$, $t_0 = 4m_\pi^2$, $\tau = \zeta/t'$.



$$369.5 \times 10^{-10} \leq a_\mu^{\text{QCD}} \leq 930.2 \times 10^{-10}$$

Backup slides: Kernel Function Approximation



The exact $K(t)$ (solid line) compared to the approximate form $K_\xi(t)$ with $\xi = 0.83$ (lower dashed line) and with $\xi = 1$ (upper dotted line).

[Retrieved from Fig. 1 of arXiv: 2404.08591 [hep-ph]]

$$K(t) \approx \frac{m_\mu^2}{3t} = K_{\text{approx}}(t)$$

Backup slides: Finding coefficients $T_{n,m}$

$N_f = 4$	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$N_f = 3$	$m = 0$	$m = 1$	$m = 2$	$m = 3$
$n = 1$	1	-	-	-	$n = 1$	1	-	-	-
$n = 2$	1.52453	25/12	-	-	$n = 2$	1.63982	9/4	-	-
$n = 3$	-11.6856	9.56054	625/144	-	$n = 3$	-10.2839	11.3792	81/16	-
$n = 4$	-92.91	-56.90	36.56	$\frac{15625}{1728}$	$n = 4$	-106.896	-46.2379	47.4048	729/64

[Retrieved from arXiv: 2404.08591 [hep-ph] Table. 3]

- coefficients $T_{n,m}$ for $N_f = 4$ (left) and $N_f = 3$ (right).
- The four-loop results are given in [2].
- the five-loop coefficient $T_{4,0}$ is from [3].
- $T_{4,1}$, $T_{4,2}$, and $T_{4,3}$ are generated from the RG analysis of [2] via the four-loop ($N_f = 4$ and $N_f = 3$) $\overline{\text{MS}}$ -scheme β function.

²M. R. Ahmady et al., Phys. Rev. D 67, 034017 (2003)

³P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. 101, 012002 (2008)

Backup slides: QCD parameters used

Parameter	Value	Source
α	1/137.036	PDG2022
$\alpha_s(M_\tau)$	0.312 ± 0.015	PDG2022
$m_u(2 \text{ GeV})$	$2.16^{+0.49}_{-0.26} \text{ MeV}$	PDG2022
$m_d(2 \text{ GeV})$	$4.67^{+0.48}_{-0.17} \text{ MeV}$	PDG2022
$m_s(2 \text{ GeV})$	$(0.0934^{+0.0086}_{-0.0034}) \text{ GeV}$	PDG2022
f_π	$(0.13056 \pm 0.00019) / \sqrt{2} \text{ GeV}$	PDG2022
$m_n \langle \bar{n}n \rangle$	$-\frac{1}{2} f_\pi^2 m_\pi^2$	Phys. Rev. 175, 2195 (1968)
$m_s \langle \bar{s}s \rangle$	$r_m r_c m_n \langle \bar{n}n \rangle$	Phys. Rev. D 103, 114005 (2021)
$r_c \equiv \langle \bar{s}s \rangle / \langle \bar{n}n \rangle$	0.66 ± 0.10	Phys. Rev. D 103, 114005 (2021)
$m_s / m_n = r_m$	$27.33^{+0.67}_{-0.77}$	PDG2022
$\langle \alpha G^2 \rangle$	$(0.0649 \pm 0.0035) \text{ GeV}^4$	Nucl. Phys. A 1039, 122743 (2023)
κ	3.22 ± 0.5	Nucl. Phys. A 1039, 122743 (2023)
$\alpha_s \langle \bar{n}n \rangle^2$	$\kappa (1.8 \times 10^{-4}) \text{ GeV}^6$	Phys. Rev. D 103, 114005 (2021)
$\alpha_s \langle \bar{s}s \rangle^2$	$r_c^2 \alpha_s \langle \bar{n}n \rangle^2$	Phys. Rev. D 103, 114005 (2021)

Here, $m_n = (m_u + m_d) / 2$ and $\langle \bar{n}n \rangle = \langle \bar{u}u \rangle = \langle \bar{d}d \rangle$.

Running of α_s through flavour threshold is not taken into consideration as the effect will be overwhelmed by other uncertainties.

[See *K.G. Chetyrkin, B.A. Kniehl, M. Steinhauser, hep-ph/9708255 v2* & *T. G. Steele and V. Elias, Mod. Phys. Lett. A 13, 3151 (1998)*]

Backup slides: Charmonium and Bottomonium Supplement

Our QCD prediction for the light-quark contributions:

$$(657.0 \pm 34.8) \times 10^{-10} \leq a_{\mu}^{\text{HVP,LO}} \leq (788.4 \pm 41.8) \times 10^{-10} .$$

Supplement our bounds with charmonium and bottomonium resonance contributions of

$$a_{\mu, \bar{c}c, \bar{b}b}^{\text{HVP,LO}} = (7.93 \pm 0.19) \times 10^{-10}$$

from [A. Keshavarzi, D. Nomura, and T. Teubner, *Phys. Rev. D* 101, 014029 (2020)]

$$\Rightarrow (664.9 \pm 34.8) \times 10^{-10} \leq a_{\mu}^{\text{HVP,LO}} \leq (796.3 \pm 41.8) \times 10^{-10}$$