Bjorken polarised sum rule with IR-finite QCD coupling

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Introduction

- We use the known renormalon structure for the inelastic Bjorken polarised sum rule (BSR) $\overline{\Gamma}_1^{\mathrm{p-n}}(Q^2)$, in order to resum the leading-twist ($D=0$) part $d(Q^2)$ of BSR approximately to all orders.
- This resummation is represented by an integral of the pQCD running coupling $a(tQ^2)$ ($\equiv \alpha_s(tQ^2)/\pi$), weighed by a characterictic function $G_d(t)$, over all momenta tQ^2. The characteristic function $G_d(t)$ is determined by the renormalon structure of $d(Q^2)$.
- The resummation (integration) is ambiguous due to the Landau singularities of the perturbative (pQCD) running coupling $a(tQ^2)$, the ambiguities reflect the renormalon structure.
- In order to avoid the resumation ambiguity, we work with various IR-finite running QCD couplings $a(tQ^2) \mapsto \mathcal{A}(tQ^2)$, i.e., they are free of Landau singularities but practically coincide with the pQCD coupling $a(tQ^2)$ at large t.
- Using truncated OPE of BSR with $D=2$ (and $D=4$) terms, we fit this expression to the data and extract the higher twist parameters $\bar{f}_2$ (and μ_6).

BSR: theoretical expressions

The (inelastic) polarised Bjorken sum rule (BSR), Γ_1^{p-n}, is the difference between the g_1 spin-dependent structure functions of the proton and nuetron integrated over the x-Bjorken range $0 < x < 1$

$$\overline{\Gamma}_1^{p-n}(Q^2) = \int_0^{1-0} dx \left[g_1^p(x, Q^2) - g_1^n(x, Q^2) \right] \ . \tag{1}$$

This quantity has been extracted at various values of $Q^2 \equiv -q^2 > 0$ from various experiments (Jefferson Lab, SLAC, COMPASS, HERMES). The theoretical Operator Product Expansion (OPE) for this quantity, truncated at $D = 4$, has the form (Bjorken, 1966, 1970)

$$\Gamma_1^{p-n,\mathrm{OPE}}(Q^2) = \left| \frac{g_A}{g_V} \right| \frac{1}{6} (1 - d(Q^2) - \delta d_{m_c}(Q^2)) + \sum_{i=2}^{\infty} \frac{\mu_{2i}(Q^2)}{Q^{2i-2}} \ , \tag{2}$$

where $|g_A/g_V|$ is the ratio of the nucleon axial charge, $d(Q^2) = a(Q^2) + \mathcal{O}(a^2)$ is the canonical pQCD part, and μ_{2i}/Q^{2i-2} are $D \geq 2$ contributions. The term proportional to $|g_A/g_V|$ is the total leading-twist (LT) contribution.

BSR: theoretical expressions

The canonical $D = 0$ part, $d(Q^2)$, has known perturbation expansion up to $a(Q^2)^4$ [where: $a(Q^2) \equiv \alpha_s(Q^2)/\pi$] (Gorishnii and Larin,1986; Larin and Vermaseren, 1991; Baikov, Chetyrkin and Kühn, 2010)

$$d(Q^2)_{\rm pt} \quad = \quad \bar{a} + \bar{d}_1 \bar{a}^2 + \bar{d}_2 \bar{a}^3 + \bar{d}_3 \bar{a}^4 + \mathcal{O}(\bar{a}^5), \tag{3}$$

where $\bar{a} = a^{\overline{\rm MS}}(Q^2)$. In any other scheme, the series in then also known up to a^4.

The next coefficient $\bar{d}_4$ can be estimated, for example, by the ECH method (Kataev and Starshenko,1995), and gives

$$\bar{d}_4 = \bar{d}_4(ECH) \pm 103. \approx 1557. \pm 32.8 \tag{4}$$

The term $\delta d(Q^2)_{\rm m_c}$ is a (small) correction due to nondecoupling of the charm quark (Blümlein et al., 2016).

BSR: theoretical expressions

The $D = 2$ coefficient μ_4 has known Q^2-dependence:

$$\mu_4(Q^2) = \bar{a}_2^{p-n} a(Q^2)^{B_{a2}} + 4\bar{d}_2^{p-n} a(Q^2)^{B_{d2}} + \bar{f}_2 a(Q^2)^{k_1}, \tag{5}$$

where the constants $\bar{a}_2^{p-n}$ and $\bar{d}_2^{p-n}$ are approximately known (Bali et al.(RQCD), 2020, 2021), and $B_{a2} = 50/81$, $B_{d2} = 77/81$, $k_1 = 32/81$ are the anomalous dimensions of the corresponding operators (Bali et al., 2020, 2021; Kawamura et al., 1996).

Renormalon structure of $d(Q^2)$

For the renormalon structure and its characteristic function, it is important to construct first the auxiliary quantity $\widetilde{d}(Q^2)$ [auxiliary to $d(Q^2)$], according to the following approach (G.C.,2018): first the above power series is reorganised in logarithmic derivatives

$$\widetilde{a}_{n+1}(Q^2) \equiv \frac{(-1)^n}{n! \beta_0^n} \left(\frac{d}{d \ln Q^2} \right)^n a(Q^2) \qquad (n = 0, 1, 2, \ldots), \tag{6}$$

[note: $\widetilde{a}_{n+1}(Q^2) = a(Q^2)^{n+1} + \mathcal{O}(a^{n+2})$] and we obtain

$$d(Q^2) = a(Q^2) + \widetilde{d}_1 \, \widetilde{a}_2(Q^2) + \ldots + \widetilde{d}_n \, \widetilde{a}_{n+1}(Q^2) + \ldots . \tag{7}$$

By the use of the ($\overline{\mathrm{MS}}$) RGE, we can relate the new coefficients $\widetilde{d}_n$ with the original ones $d_n, d_{n-1}, \ldots$.

Renormalon structure of $d(Q^2)$

Using these modified coefficients $\widetilde{d}_n$, the modified Borel transform is

$$\mathcal{B}[\widetilde{d}](u) \equiv 1 + \frac{\widetilde{d}_1}{1!\beta_0} u + \ldots + \frac{\widetilde{d}_n}{n!\beta_0^n} u^n + \ldots . \qquad (8)$$

It turns out that this transform has the simple one-loop-type renormalisation scale dependence (in contrast to the original $\mathcal{B}[d](u)$)

$$\frac{d}{d \ln \kappa} \widetilde{d}_n(\kappa) = n\beta_0 \widetilde{d}_{n-1}(\kappa) \quad \Rightarrow \quad \mathcal{B}[\widetilde{d}](u;\kappa) = \kappa^u \mathcal{B}[\widetilde{d}](u), \qquad (9)$$

where $\kappa \equiv \mu^2/Q^2$.

Renormalon structure of $d(Q^2)$

As a consequence, it can be shown (G.C., 2018) that this Borel transform $\mathcal{B}[\widetilde{d}](u)$ has a structure very similar to the known large-β_0 structure of the Borel $\mathcal{B}[d](u)$

$$\mathcal{B}[\widetilde{d}](u) = \exp\left(\widetilde{K} u\right) \pi \Big\{ \widetilde{d}_1^{\rm IR} \frac{1}{(1-u)^{\kappa_1}} + \widetilde{d}_2^{\rm IR} \frac{1}{(2-u)} + \widetilde{d}_1^{\rm UV} \frac{1}{(1+u)} + \widetilde{d}_2^{\rm UV} \frac{1}{(2+u)} \Big\}, \quad (10)$$

Here, $\kappa_1 = 1 - k_1$, where $k_1 = 32/81$ is the aforementioned anomalous dimension of the $D = 2$ OPE term. The five parameters ($\widetilde{K}$ and the residues $\widetilde{d}_1^{\rm IR}, \widetilde{d}_2^{\rm IR}, \widetilde{d}_1^{\rm UV}, \widetilde{d}_2^{\rm UV}$) are determined by the knowledge of the first five coefficients d_n (and thus $\widetilde{d}_n$), $n = 0, 1, 2, 3, 4$.

Renormalon structure of $d(Q^2)$

It can be shown (G.C., 2018; C.A.,G.C. and D.Teca, 2023) that this ansatz for $\mathcal{B}[\widetilde{d}](u)$ implies the theoretically expected structure of the corresponding rernormalon terms in the Borel $\mathcal{B}[d](u)$ of the canonical BSR $d(Q^2)$

$$\begin{aligned} \mathcal{B}[\widetilde{d}](u) &= \frac{A}{(p \mp u)^{\kappa}} \Rightarrow \\ \mathcal{B}[d](u) &= \frac{B}{(p \mp u)^{\kappa \pm p\beta_1/\beta_0^2}} \left[1 + \mathcal{O}(p \mp u)\right] \end{aligned} \tag{11}$$

where β_0 and β_1 are the one-loop and two-loop QCD β-coefficients

$$\begin{aligned} \frac{d\, a(Q^2)}{d \ln Q^2} &= -\beta_0 a(Q^2)^2 - \beta_1 a(Q^2)^3 - \beta_2 a(Q^2)^4 - \ldots \\ &= -\beta_0 a(Q^2)^2 \left[1 + c_1 a(Q^2) + c_2 a(Q^2)^2 + \ldots\right]. \end{aligned}$$

Renormalon structure of $d(Q^2)$

We use P44-type β-functions:

$$\frac{d a(Q^2)}{d \ln Q^2} = \beta(a(Q^2)))_{\rm P44}$$

$$\equiv -\beta_0 a(Q^2)^2 \frac{\left[1 + \alpha_0 c_1 a(Q^2) + \alpha_1 c_1^2 a(Q^2)^2\right]}{\left[1 - \alpha_1 c_1^2 a(Q^2)^2\right] \left[1 + (\alpha_0 - 1) c_1 a(Q^2) + \alpha_1 c_1^2 a(Q^2)^2\right]} ,$$

where $c_j \equiv \beta_j/\beta_0$ and

$$\alpha_0 = 1 + \sqrt{c_3/c_1^3}, \quad \alpha_1 = c_2/c_1^2 + \sqrt{c_3/c_1^3}.$$

Expansion gives:

$$\beta(a(Q^2))_{\rm P44} = -\beta_0 a(Q^2)^2 \left[1 + c_1 a(Q^2) + c_2 a(Q^2)^2 + c_3 a(Q^2)^3 \ldots\right].$$

Renormalon structure of $d(Q^2)$

For these P44-type β-functions we have explicit solutions (G.C., Kondrashuk, 2011):

$$a(Q^2) = \frac{2}{c_1}\left[-\sqrt{\omega_2} - 1 - W_{\mp 1}(z) + \sqrt{(\sqrt{\omega_2}+1+W_{\mp 1}(z))^2 - 4(\omega_1+\sqrt{\omega_2})}\right]^{-1},$$

where $\omega_1 = c_2/c_1^2$, $\omega_2 = c_3/c_1^3$, $Q^2 = |Q^2| \exp(i\phi)$, and $W_{\mp 1}(z)$ are two branches of the Lambert function. When $0 \le \phi < \pi$, $W_{-1}(z)$ is used; when $-\pi \le \phi < 0$, $W_{+1}(z)$ is used. The argument $z = z(Q^2)$ appearing in $W_{\pm 1}(z)$ is

$$z \equiv z(Q^2) = -\frac{1}{c_1 e}\left(\frac{\Lambda_L^2}{Q^2}\right)^{\beta_0/c_1}.$$

We call Λ_L Lambert scale ($\Lambda_L^2 \sim 0.1 \ \mathrm{GeV}^2$).

Renormalon structure of $d(Q^2)$

Table: The values of $\widetilde{K}$ and of the renormalon residues $\widetilde{d}_{i,j}^{\rm X}$ (X=IR,UV) for the five-parameter ansatz (10) in the (5-loop) $\overline{\rm MS}$ and various P44-type schemes: $\overline{\rm MS}$ P44 ($c_2 = 4.471$ and $c_3 = 20.990$); ${\rm P44}^{(0)}$ ($c_2 = 6.$ and $c_3 = 20.$); LMM P44 ($c_2 = 9.297$ and $c_3 = 71.454$). Here, d_4 is taken such that $d_4(\overline{\rm MS}) = 1557.43$ as predicted by ECH. The last line is for the 5-loop $\overline{\rm MS}$ with $d_4(\overline{\rm MS}) = 1557.43 - 32.84 = 1524.59$, cf. Eq. (4).

scheme	$\widetilde{K}$	$\widetilde{d}_1^{\rm IR}$	$\widetilde{d}_2^{\rm IR}$	$\widetilde{d}_1^{\rm UV}$	$\widetilde{d}_2^{\rm UV}$
$\overline{\rm MS}$	-1.82336	7.8156	-14.8199	-0.0413348	-0.0920349
$\overline{\rm MS}$ P44	-1.83223	7.86652	-14.9299	-0.0444416	-0.0776748
${\rm P44}^{(0)}$	0.450041	0.331813	0.231437	-0.0809782	-0.0964868
LMM P44	0.443075	0.0726314	0.880932	-0.00867699	-0.372221

Renormalon structure of $d(Q^2)$

Table: The coefficients d_n for the case of $\overline{\mathrm{MS}}$ ($c_2 = 4.471$ and $c_3 = 20.990$), $\mathrm{P44}^{(0)}$ ($c_2 = 6.$ and $c_3 = 20.$) and LMM P44 ($c_2 = 9.297$ and $c_3 = 71.454$). $N_f = 3$ taken throughout.

n	$d_n(\overline{\mathrm{MS}})$	$d_n(\mathrm{P44}^{(0)})$	$d_n(\mathrm{LMM})$
0	1	1	1
1	3.58333	3.58333	3.58333
2	20.2153	15.6863	15.3893
3	175.749	143.787	115.932
4	1557.43	1243.48	1074.32
5	24746.	16422.2	10691.9
6	315887.	208828.	120500.
7	6.0460×10^6	3.65668×10^6	1.62183×10^6
8	1.06009×10^8	6.32015×10^7	2.48533×10^7
9	2.44469×10^9	1.4051×10^9	4.54306×10^8
10	5.38798×10^{10}	3.07916×10^{10}	9.40672×10^9

Renormalon structure of $d(Q^2)$

The characteristic function $G_d(t)$ that enters in the (formal) resummation of the (canonical BSR) $d(Q^2)$

$$d(Q^2)_{\rm res} = \int_0^{\infty} \frac{dt}{t} G_d(t) a(t e^{-\widetilde{K}} Q^2) \qquad (12)$$

is the inverse transform of the Borel $\mathcal{B}[\widetilde{d}](u)$ (G.C., 2018)

$$G_d(t) = \frac{1}{2\pi i} \int_{u_0 - i\infty}^{u_0 + i\infty} du {\rm B}[\widetilde{d}](u) t^u, \qquad (13)$$

where $-1 < u_0 < +1$. Explicit evaluation gives:

$$\begin{aligned} G_d(t) &= \Theta(1-t)\pi \left[\frac{\widetilde{d}_1^{\rm IR} t}{\Gamma(1-k_1) \ln^{k_1}(1/t)} + \widetilde{d}_2^{\rm IR} t^2 \right] \\ & + \Theta(t-1)\pi \left[\frac{\widetilde{d}_1^{\rm UV}}{t} + \frac{\widetilde{d}_2^{\rm UV}}{t^2} \right]. \qquad (14) \end{aligned}$$

Renormalon structure of $d(Q^2)$

In practice, the pQCD coupling $a(Q'^2)$ has (unphysical) Landau singularities at low positive Q'^2, thus the integration (12) has to avoid them, which we do with a PV-type of regularisation

$$d(Q^2)_{\rm res} = Re\left[\int_0^\infty \frac{dt}{t} G_d(t) a(t e^{-\widetilde{K}} Q^2 + i\varepsilon)\right] \qquad (15)$$

On the other hand, if we use an IR-safe coupling $a(Q^2) \mapsto \mathcal{A}(Q^2)$ that has no Landau singularities but practically coincides with $a(Q^2)$ at large $|Q^2| > \Lambda^2_{\rm QCD}$, such as 3δQCD coupling (C.A., G.C. et al., 2017), no additional regularisation is needed

$$d(Q^2)_{\rm res} = Re\left[\int_0^\infty \frac{dt}{t} G_d(t) \mathcal{A}(t e^{-\widetilde{K}} Q^2)\right] \qquad (16)$$

IR-finite couplings $\mathcal{A}(Q^2)$

The spectral function $\rho^{(\mathrm{pt})}(\sigma)$ of pQCD coupling $a(Q^2)$

$$\rho^{(\mathrm{pt})}(\sigma) \equiv \mathrm{Im}\, a(-\sigma - i\epsilon), \tag{17}$$

has nonzero values even for $\sigma < 0$, on the Landau cut ($\Lambda_L^2 \leq \sigma < 0$). If, on the other hand, we define the spectral function without such a cut

$$\rho_{\mathcal{A}}^{(n\delta)}(\sigma) = \pi \sum_{j=1}^{n} \mathcal{F}_j \, \delta(\sigma - M_j^2) + \Theta(\sigma - M_0^2)\rho^{(\mathrm{pt})}(\sigma) \ , \tag{18}$$

we obtain coupling $\mathcal{A}(Q^2)$ that is free of the Landau singularities and is finite at $Q^2 \to 0$

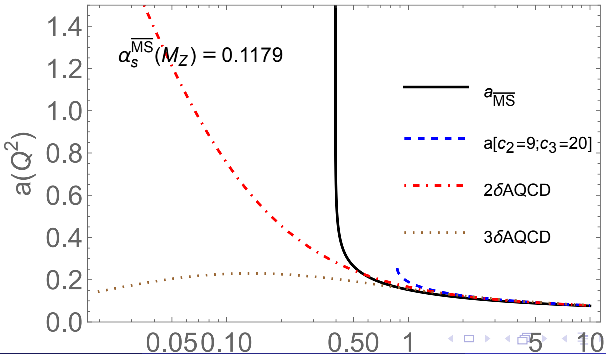
$$\mathcal{A}^{(n\delta)}(Q^2) \left(\equiv \frac{1}{\pi} \int_0^\infty d\sigma \frac{\rho_{\mathcal{A}}(\sigma)}{(\sigma + Q^2)} \right) = \sum_{j=1}^{n} \frac{\mathcal{F}_j}{(Q^2 + M_j^2)} + \frac{1}{\pi} \int_{M_0^2}^\infty d\sigma \frac{\rho_1^{(\mathrm{pt})}(\sigma)}{(Q^2 + \sigma)} \ .$$

IR-finite couplings $\mathcal{A}(Q^2)$

We can impose that $\mathcal{A}(Q^2)$ practically coincides with the (underlying) pQCD coupling $a(Q^2)$ at high Q^2

$$\mathcal{A}(Q^2) - a(Q^2) \sim \left(\frac{\Lambda_L^2}{Q^2} \right)^5 \qquad (|Q^2| > \Lambda_L^2), \tag{19}$$

and at low Q^2 we can impose additional physically motivated conditions. This determines then the coupling.



$d(Q^2)$ with pQCD and IR-finite couplings $\mathcal{A}(Q^2)$

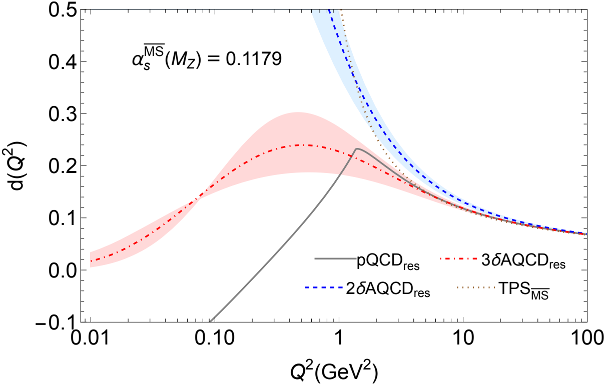


Figure: $d(Q^2)_{\rm res}$ (at $N_f = 3$), according to Eq. (16), for the 2δACD and 3δAQCD coupling ('$2\delta\mathrm{AQCD}_{\rm res}$', '$3\delta\mathrm{AQCD}_{\rm res}$'), with three hadronic threshold scales $M_1 = 0.150^{+0.100}_{-0.050}$ GeV of their spectral function. For comparison, $d(Q^2)_{\rm res}$ for the pQCD coupling (underlying to 2δACD ('$\mathrm{pQCD}_{\rm res}$'). Included is the truncated perturbation series (TPS) in powers of $a = a(Q^2)$ in the $\overline{\rm MS}$ scheme up to $\sim a^5$ ('$\mathrm{TPS}_{\overline{\rm MS}}$'). We use throughout $\alpha_s^{\overline{\rm MS}}(M_Z^2; N_f = 5) = 0.1179$ and $d_4^{\overline{\rm MS}} = 1557.4$.

Fitting the BSR data, extraction of α_s

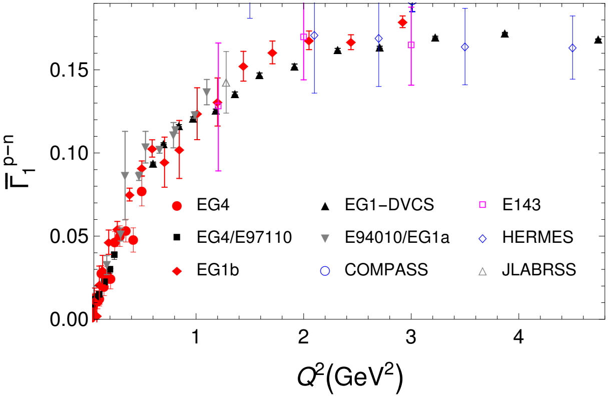


Figure: The measured results for the inelastic BSR $\overline{\Gamma}_1^{p-n}(Q^2)$ for different experiments, with the *statistical* uncertainties.

Fitting the BSR data, extraction of α_s

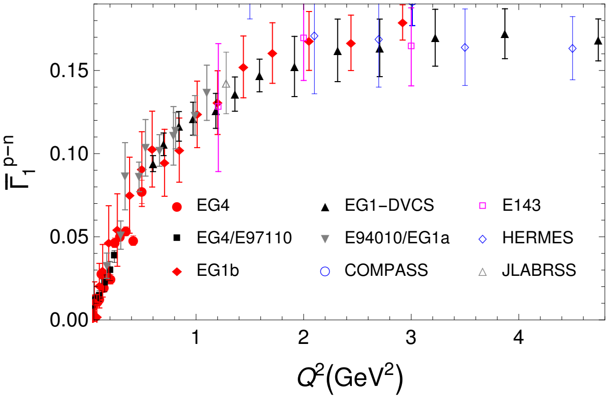


Figure: The measured results for the inelastic BSR $\overline{\Gamma}_1^{p-n}(Q^2)$ for different experiments, with the *systematic* uncertainties.

Fitting the BSR data, extraction of α_s

$$\chi^2(j_{\rm min}; k) = \frac{1}{(j_{\rm max} - j_{\rm min} + 1 - N_{\rm p})} \sum_{j=j_{\rm min}}^{j_{\rm max}} \frac{\left[\overline{\Gamma}_1^{\rm p-n,OPE}(Q_j^2) - \overline{\Gamma}_1^{\rm p-n}(Q_j^2)_{\rm exp} \right]^2}{\sigma(Q_j^2; k)^2}$$

$$\sigma^2(Q_j^2; k) = \sigma^2_{\rm stat}(Q_j^2) + k \, \sigma^2_{\rm sys}(Q_j^2).$$

k is adjusted until ${\rm Min}\, \chi^2(j_{\rm min}; k) = 1$. Then $\sigma(Q_j^2; k) = \sigma(Q_j)_{\rm uncorr.}$ and $(1 - \sqrt{k})\sigma_{\rm sys}(Q_j^2) = \sigma(Q_j)_{\rm corr.}$.

Fitting the BSR data, extraction of α_s

The fit with data, for $\alpha_s^{\overline{\rm MS}}(M_Z^2) = 0.1179$ and $Q^2_{\rm min} \approx 0.6 \ {\rm GeV}^2$, gives $k \approx 0.130, 0.158$ in 2δ and 3δAQCD, and:

$$\bar{f}_2^{(2\delta)} = -0.0082^{+0.0180}_{-0.0196}(c_2 c_3)^{+0.0283}_{-0.0274}(\alpha_s)^{-0.0001}_{+0.0003}(d_4)^{-0.0763}_{+0.0541}(M_1)^{-0.0209}_{+0.0210}(a_2 d_2)$$
$$^{-0.0006}_{+0.0035}(Q^2_{\rm min}) \pm 0.0038({\rm exp.u.}) \pm 0.0207({\rm exp.c.}) \pm 0.0001(k).$$

$$\bar{f}_2^{(3\delta)} = -0.2534^{+0.0110}_{-0.0109}(\alpha_s) \pm 0.0001(d_4)^{+0.0447}_{-0.0427}(M_1)^{-0.0187}_{+0.0186}(a_2 d_2)$$
$$^{-0.0044}_{+0.0019}(Q^2_{\rm min}) \pm 0.0042({\rm exp.u.}) \pm 0.0202({\rm exp.c.}) \mp 0.0013(k).$$

In pQCD $Q^2_{\rm min} \approx 1.7 \ {\rm GeV}^2$

$$\bar{f}_2^{(\rm pQCD)} = -0.107^{+0.022}_{-0.030}(c_2 c_3) \pm 0.020(\alpha_s) \mp 0.009(d_4) \mp 0.067({\rm ren})$$
$$^{+0.012}_{-0.029}(Q^2_{\rm min}) \pm 0.033({\rm exp.u.}) \pm 0.154({\rm exp.c.}),$$

Similar results are obtained for two-parameter fit ($\bar{f}_2$ and μ_6). When using pQCD coupling, much higher $Q^2_{\rm min} \approx 1.7 \ {\rm GeV}^2$ is needed, and experimental uncertainties of the extracted values are one order of magnitude larger: $\pm 0.033({\rm exp.u.})$ and $\pm 0.154({\rm exp.c.})$.

Fitting the BSR data, extraction of α_s

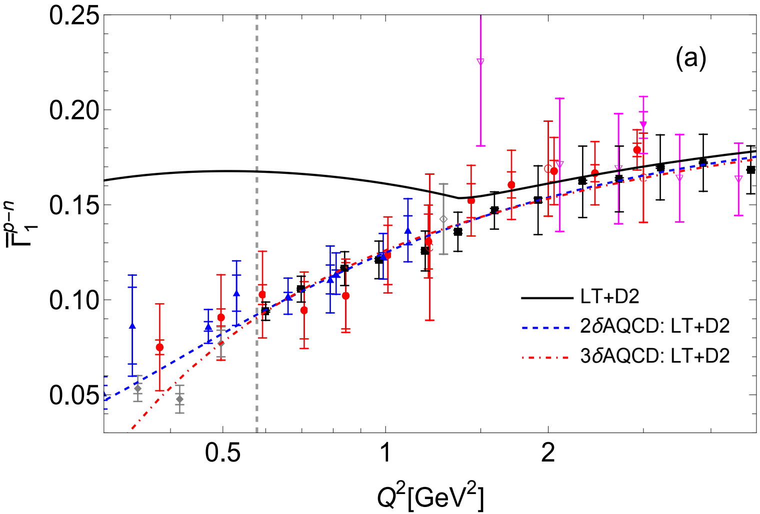


Figure: Results of fitting of BSR $\overline{\Gamma}_1^{\mathrm{p-n}}(Q^2)$, with one-parameter fit. Experimental data, and the corresponding pQCD fit (solid line) are included.

Fitting the BSR data, extraction of α_s

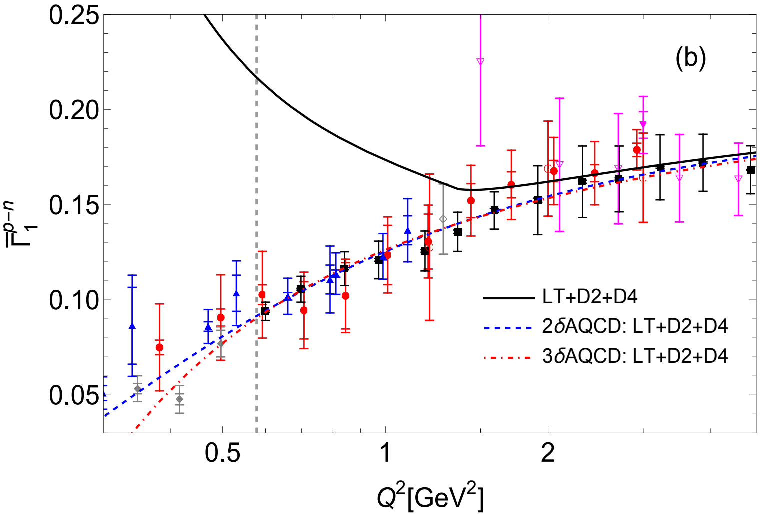


Figure: Results of fitting of BSR $\overline{\Gamma}_1^{\mathrm{p-n}}(Q^2)$, with two-parameter fit. Experimental data, and the corresponding pQCD fit (solid line) are included.

Conclusions

- From the knowledge of the pQCD expansion coefficients $\widetilde{d}_n$ of canonical part $d(Q^2)$ BSR, we determined the parameters of the Borel transform $\mathcal{B}[\widetilde{d}](u)$.
- Inverse Mellin transform of $\mathcal{B}[\widetilde{d}](u)$ gave us the characteristic function $G_d(t)$ which allowed us to resum/evaluate $d(Q^2)$ in BSR $\Gamma_1^{p-n}(Q^2)$. When using IR-finite QCD coupling, no regularisation is needed. When using pQCD coupling, a (PV-type of) regularisation is needed.
- When using IR-finite couplings, we can extract the higher-twist parameters of BSR ($\bar{f}_2$, or both $\bar{f}_2, \mu_6$) with relatively small experimental uncertainties. Those uncertainties become much higher when pQCD coupling is used, and the Q^2-interval of fit has to be significantly reduced as well.
- A reliable extraction of the value of $\alpha_s^{\overline{\rm MS}}(M_Z^2)$ cannot be made, principally because of the present large experimental uncertainties of the BSR data. For the central input values we obtain in 2δ and 3δAQCD the minimal k and thus the preferred values of $\alpha_s^{\overline{\rm MS}}(M_Z^2)$ for $\alpha_s^{\overline{\rm MS}}(M_Z^2) \approx 0.117$ when $Q^2_{\rm min} \approx 0.6 \ {\rm GeV}^2$, but the minimum disappears when $Q^2_{\rm min}$ increases.