Nonperturbative uncertainties in $\alpha_s(m_\tau)$

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In collaboration with
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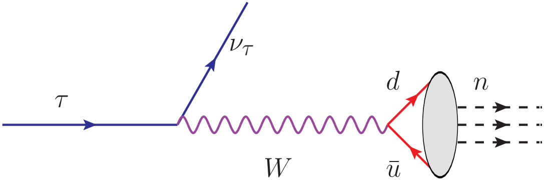
The hadronic decay of the τ

$$\tau^- \to n + \nu_\tau, \quad n = \pi^-, \pi^-\pi^0, K^-\pi^0, \cdots$$



$$\mathcal{L}_{SM} \sim W^\mu J^l_\mu + W^\mu J^q_\mu, \quad J^l \sim \bar{\nu}_\tau \Gamma \tau,\ J^q \sim V_{uD} \bar{D} \Gamma u$$

$$q^2 \ll M_W^2 \sim v^2 \to \mathcal{L}^{eff}_{SM} \sim \frac{1}{v^2} J^q \cdot J^l$$

$$\frac{d\Gamma^{(n)}}{dq^2} \sim \sum \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \rho^{(n,1)}(s) + \rho^{(n,0)}(s)\right]$$

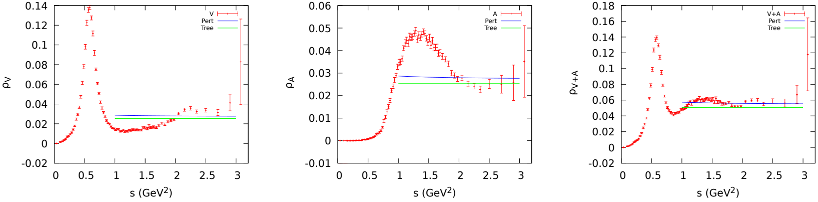
$$\rho^{(n)}(q^2) \sim \int d\phi_m \langle n|J|0\rangle\langle 0|J^\dagger|n\rangle \text{ e.g. } \rho^{(\pi,0)} \sim f_\pi^2\, \delta^4\left(q^2 - m_\pi^2\right)$$

$$\frac{d\Gamma^{(n)}}{dq^2} \sim \sum \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \rho^{(n,1)}(s) + \rho^{(n,0)}(s)\right]$$

$$\rho^{(n)}(q^2) \sim \int d\phi_m \langle n|J|0\rangle\langle 0|J^\dagger|n\rangle \text{ e.g. } \rho^{(\pi,0)} \sim f_\pi^2\,\delta^4\left(q^2 - m_\pi^2\right)$$

In general $\rho^{(n)}(q^2)$ poorly known. However $\sum_{n_{V,A}} \rho^{V,A}(q^2) \to \mathrm{Im}\Pi_{V,A}(q^2)$

$$\Pi_{V/A}(q^2) \sim \int d^4x\, e^{-iqx} \langle T(J_{V/A}(x)J_{V/A}(0))\rangle$$



ALEPH

Precision physics with inclusive tau decays

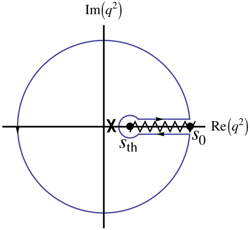
Experimental access to $\rho_{V,A} \sim \mathrm{Im}\Pi_{V,A}(q^2)$

$$\Pi_{V/A}(q^2) \sim \int d^4x\, e^{-iqx} \langle T(J_{V/A}(x) J_{V/A}(0)) \rangle$$

- Where do we know Π? Large-Euclidean momenta $\to$ OPE. pQCD plus:

$$\Pi_J^{\mathrm{OPE}}(s)\Big|_{D>0} = \sum_{D>0} \frac{\mathcal{O}_{D,J}(\mu) + \mathcal{P}_{D,J} \ln(-s/\mu^2)}{(-s)^{D/2}}, \qquad \mathcal{O}_D, \mathcal{P}_D \sim \alpha_s \mathcal{O}_D \text{ SVZ '79}$$

- What else do we know? Analyticity



$$A_J^{(n)}(s_0) \equiv \underbrace{\pi \int_{s_{\rm th}}^{s_0} \frac{ds}{s_0} \left(\frac{s}{s_0}\right)^n \rho_J(s)}_{\text{Experiment}} = \underbrace{\frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0} \left(\frac{s}{s_0}\right)^n \Pi_J(s)}_{\sim\text{OPE}}$$

$\sim$ BNP'92

The inclusive hadronic decay of the τ

$$A^{(n)}(s_0) \equiv \pi \int_{s_{\rm th}}^{s_0} \frac{ds}{s_0} \left(\frac{s}{s_0}\right)^n \rho_J(s) = \frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0} \left(\frac{s}{s_0}\right)^n \Pi_J(s)$$

$$A^{(n)}(s_0) = A^{(n)}_{\rm pert}(s_0) + A^{(n),{\rm OPE}}_{\rm D>0}(s_0) + \Delta A^{(n),{\rm DV}}(s_0)$$

Perturbative

$$A^{(n)}_{\rm pert}(s_0) = \frac{1}{8\pi^2(n+1)} \sum_m K_m \int_{-\pi}^{\pi} d\varphi \, \left(1-(-1)^{n+1} e^{i\varphi(n+1)}\right) a_s^m \left(s_0 e^{i\varphi}\right)$$

Power corrections

$$A^{(n)}(s_0)\Big|_{D>0} = -\pi \sum_{p=2} \frac{d_p^{(n)}}{(-s_0)^p}\,, \quad d_p^{(n)} = \begin{cases} \mathcal{O}_{2p}(s_0), & \text{if } p = n+1 \\ \dfrac{\mathcal{P}_{2p}}{n-p+1}, & \text{if } p \neq n+1 \end{cases}$$

Duality Violations (DV)

$$\Delta A^{\omega}(s_0) \equiv \tfrac{i}{2} \oint_{|s|=s_0} \tfrac{ds}{s_0}\, \omega(s) \left\{\Pi(s) - \Pi^{\rm OPE}(s)\right\} = -\pi \int_{s_0}^{\infty} \tfrac{ds}{s_0}\, \omega(s)\, \Delta\rho^{\rm DV}(s)$$

Uncertainties in α_s. Generalities

$$A^{(n)}(s_0)[\alpha_s, K_{m\geq 5}, \beta_{m\geq 6}, \mathcal{O}_{2n+2}(\mu), \mathcal{P}_{D\neq 2n+2}, \Delta A^{(n)}(s_0)]$$

- Perturbative uncertainties: variation in K_5 and scale
- Power corrections

$$A^{(n)}(s_0)\Big|_{D>0} = -\pi \sum_{p=2} \frac{d_p^{(n)}}{(-s_0)^p}, \quad d_p^{(n)} = \begin{cases} \mathcal{O}_{2p}(s_0), & \text{if } p = n+1 \\ \dfrac{\mathcal{P}_{2p}}{n-p+1}, & \text{if } p \neq n+1 \end{cases}$$

  - They exist beyond perturbation theory. E.g. $\mathcal{O}_{6,V-A} \sim -(0.003, 0.004)\,\mathrm{GeV}^6$
  - Tiny, but poorly known
  - Ideally go to high energy
  - For fixed energy, ideally suppress lower dimensions and $\mathcal{O}_D$ with respect to $\mathcal{P}_D$
- Duality violations

$$\Delta A^{\omega}(s_0) = -\pi \int_{s_0}^{\infty} \tfrac{ds}{s_0}\, \omega(s)\, \Delta\rho^{\mathrm{DV}}(s)$$

  - They clearly exist (OPE does not fully describe observed spectral function)
  - Expected (and at least partially observed) to go to zero very fast
  - Also tiny (in integrals) but poorly known
  - Ideally go to high energies and reduce the contribution near s_0

Extraction on α_s, ALEPH-like weights

$$A^{(n)}(s_0)[\alpha_s, K_{m\geq 5}, \beta_{m\geq 6}, \mathcal{O}_{2n+2}(\mu), \mathcal{P}_{D\neq 2n+2}, \Delta A^{(n)}(s_0)]$$

- ALEPH set of weights at m_τ:
  $\omega_{kl}(x) = (1-x)^{2+k}\, x^l\, (1+2x)\,, \quad (k,l) = \{(0,0),\, (1,0), (1,1), (1,2), (1,3)\}$
- Fitted parameters: $\alpha_s, \mathcal{O}_4, \mathcal{O}_6, \mathcal{O}_8$

Rationale behind the choice/assumptions:

- Double zero at $s_0 = m_\tau^2$ should be enough to discriminate poorly known DVs
- Assume energy is high enough so that $\mathcal{O}_{D>8}$ and $\mathcal{P}_{D>4}$ can be neglected

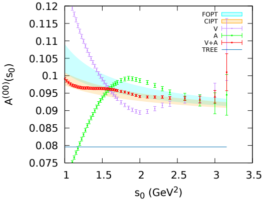
Good quality fits and consistent results. But yet some potential weaknesses

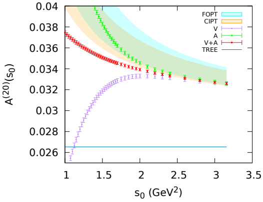
- Truncation choice somewhat ambiguous
- High-dimensional contributions indirectly enhanced by long prefactors

First estimation attempt: incorporate $\mathcal{O}_{D=10}$ and add the difference
More tests: change sets of weights reducing previous weaknesses. Same $\alpha_s(m_\tau)$

More illustrative tests

- Reliability of $\alpha_s(m_\tau)$ requires nonperturbative corrections to be small
- Let us ignore ALL nonperturbative corrections





Weight (n, m)	$\alpha_s(m_\tau^2)$ FOPT	$\alpha_s(m_\tau^2)$ CIPT	Weight (n, m)	$\alpha_s(m_\tau^2)$ FOPT	$\alpha_s(m_\tau^2)$ CIPT
(1,0)	$0.315\,^{+0.012}_{-0.007}$	$0.327\,^{+0.012}_{-0.009}$	(2,0)	$0.311\,^{+0.015}_{-0.011}$	$0.314\,^{+0.013}_{-0.009}$
(1,1)	$0.319\,^{+0.010}_{-0.006}$	$0.340\,^{+0.011}_{-0.009}$	(2,1)	$0.311\,^{+0.011}_{-0.006}$	$0.333\,^{+0.009}_{-0.007}$
(1,2)	$0.322\,^{+0.010}_{-0.008}$	$0.343\,^{+0.012}_{-0.010}$	(2,2)	$0.316\,^{+0.010}_{-0.005}$	$0.336\,^{+0.011}_{-0.009}$
(1,3)	$0.324\,^{+0.011}_{-0.010}$	$0.345\,^{+0.013}_{-0.011}$	(2,3)	$0.318\,^{+0.010}_{-0.006}$	$0.339\,^{+0.011}_{-0.008}$
(1,4)	$0.326\,^{+0.011}_{-0.011}$	$0.347\,^{+0.013}_{-0.012}$	(2,4)	$0.319\,^{+0.009}_{-0.007}$	$0.340\,^{+0.011}_{-0.009}$
(1,5)	$0.327\,^{+0.015}_{-0.013}$	$0.348\,^{+0.014}_{-0.012}$	(2,5)	$0.320\,^{+0.010}_{-0.008}$	$0.341\,^{+0.011}_{-0.009}$

- Similar α_s independently on the weight at m_τ^2
- Similar α_s at all channels at m_τ^2
- Most inclusive channel: similar α_s at any $s_0 < m_\tau^2$

Other included approaches

- Fit to the s_0-dependence of fix weights. For example: $\left(1 - \frac{s}{s_0}\right)^2$
  - Advantage: one can include all $\mathcal{O}_D$ in the fit
  - Disadvantage: more exposed to unknown DVs
  - Obtained $\alpha_s(m_\tau^2)$ stable in $V + A$ channel. Add s_0-fluctuations as DV estimate
- Add $e^{-a\frac{s}{s_0}}$ factors to the weights
  - It reduces high-energy tail (DV) contribution
  - a small do not enhance unaccounted power corrections ($\frac{a}{(D/2)!}\mathcal{O}_D$ vs $\alpha_s\mathcal{O}_D$)
  - Dedicated analysis finds stability in all channels for wide intervals of a and s_0

Method	$\alpha_s^{(n_f=3)}(m_\tau^2)$		
	CIPT	FOPT	Average
$\omega_{kl}(x)$ weights	$0.339\,^{+0.019}_{-0.017}$	$0.319\,^{+0.017}_{-0.015}$	$0.329\,^{+0.020}_{-0.018}$
$\hat\omega_{kl}(x)$ weights	$0.338\,^{+0.014}_{-0.012}$	$0.319\,^{+0.013}_{-0.010}$	$0.329\,^{+0.016}_{-0.014}$
$\omega^{(2,m)}(x)$ weights	$0.336\,^{+0.018}_{-0.016}$	$0.317\,^{+0.015}_{-0.013}$	$0.326\,^{+0.018}_{-0.016}$
s_0 dependence	0.335 ± 0.014	0.323 ± 0.012	0.329 ± 0.013
$\omega_a^{(1,m)}(x)$ weights	$0.328\,^{+0.014}_{-0.013}$	$0.318\,^{+0.015}_{-0.012}$	$0.323\,^{+0.015}_{-0.013}$
Average	0.335 ± 0.013	0.320 ± 0.012	0.328 ± 0.013

Duality Violation approach

Nonperturbative corrections in $\omega = 1$

- Possibly be good from the point of view of power corrections. Yet

$$A(s_0)|_{\mathcal{P}_D} = \pi\left(-\frac{\mathcal{P}_6}{2s_0^3}+\frac{\mathcal{P}_8}{3s_0^4}-\frac{\mathcal{P}_{10}}{4s_0^5}+\frac{\mathcal{P}_{12}}{5s_0^6}-\frac{\mathcal{P}_{14}}{6s_0^7}+\frac{\mathcal{P}_{16}}{7s_0^8}+\cdots\right)$$

- Not optimal from the point of view of reducing DVs

One possibility. DV fluctuations in $V+A$ for $s_0 \in (\frac{m_\tau^2}{2}, m_\tau^2)$ have a very subleading role in integral and then α_s

- Assume fluctuations do not increase at $s_0 > m_\tau^2 \to \alpha_s(m_\tau^2)$
- V more unstable. But if one assumes at $s_0 \sim m_\tau^2$ stabilizes, one obtains same strong coupling.

Another possibility: modelling DVs Boito et al.

$$\Delta\rho^{\mathrm{DV}}(s) \;=\; \mathcal{G}(s)\, e^{-(\delta+\gamma s)}\, \sin(\alpha+\beta s) \qquad\qquad s > \hat{s}_0$$

Fit $\{\alpha_s, \delta, \gamma, \alpha, \beta\}$ to s_0 dependence of $A(s_0)$, i.e. fit $\left\{A(\hat{s}_0), \rho(\hat{s}_0 < s_0 < m_\tau^2)\right\}$

Duality Violation approach

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From ALEPH one finds at $\mathcal{G}(s) = 1, \hat{s}_0 = 1.55\,\mathrm{GeV}^2$

$$\alpha_s(m_\tau) = 0.298 \pm 0.010\,.$$

First weakness

- Argued to be better wrt standard ones because is free from unknown $\mathcal{O}_{D>10}$
- But one yet has all $\mathcal{P}_D \sim 0.2\,\mathcal{O}_D$ contributions
- In contrast to standard ones one relies on them to be negligible at $s_0 < \frac{m_\tau^2}{2}$
- Impact of neglecting the latter with respect to the former scales as $0.2 \cdot 2^{D/2}$

If power corrections of $D > 10$ are a concern at m_τ^2, avoid going below it...

Duality Violation approach

Possibility: modelling DVs Boito et al.

$$\Delta\rho^{\mathrm{DV}}(s) \;=\; \mathcal{G}(s)\, e^{-(\delta+\gamma s)}\, \sin(\alpha+\beta s) \qquad\qquad s > \hat{s}_0$$

Fit $\{\alpha_s, \delta, \gamma, \alpha, \beta\}$ to s_0 dependence of $A(s_0)$, i.e. fit $\left\{A(\hat{s}_0), \rho(\hat{s}_0 < s_0 < m_\tau^2)\right\}$
From ALEPH one finds at $\mathcal{G}(s) = 1, \hat{s}_0 = 1.55\,\mathrm{GeV}^2$

$$\alpha_s(m_\tau) = 0.298 \pm 0.010\,.$$

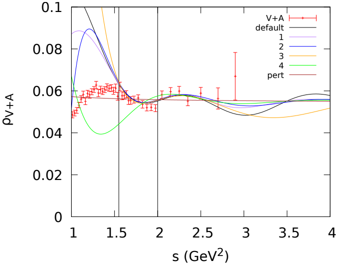
Second weakness. Not real motivation for $\mathcal{G}(s) = 1, \hat{s}_0 = 1.55\,\mathrm{GeV}^2$

1. $\mathcal{G}_V(s) = s^8, \qquad \hat{s}_0 = 1.55.$
2. $\mathcal{G}_V(s) = 1 - \dfrac{1.35}{s}, \qquad \hat{s}_0 = 1.55.$
3. $\mathcal{G}_V(s) = 1 - \dfrac{2}{s}, \qquad \hat{s}_0 = 1.55.$
4. $\mathcal{G}_V(s) = 1, \qquad \hat{s}_0 = 2, \qquad \alpha_s = 0.320.$

Variation	$\alpha_s(m_\tau^2)$	δ_V	γ_V	α_V	β_V	p-value (%)
Default	0.298	3.6	0.6	-2.3	4.3	5.3
1	0.314	1.0	4.6	-1.5	3.9	7.7
2	0.319	-0.19	1.8	-0.8	3.5	7.8
3	0.260	0.23	1.2	3.2	2.1	6.4
4	0.320	0.56	1.9	0.15	3.1	6.9

Test standard assumptions with DV models

Let us take $\alpha_s^{\rm FOPT} \in (0.26 - 0.32)$ suplemented by DV parameters and check $V + A$

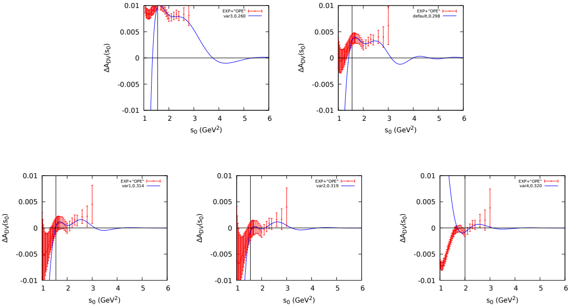


- Convergence of data to models at assumed point $\hat{s}_0$ much worse than convergence of data to OPE itself
- Agreement of models in fitted regions
- Complete disagreement above fitted regions (fully explains α_s splitting)
- Small α_s models display DVs larger than a_1 resonance at $s_0 > m_\tau$

Test standard assumptions with DV models: $\omega = 1$

$$A^{(0)}(s_0) = A^{(0)}_{\mathrm{pert}}(s_0) + \Delta A^{(0),\mathrm{DV}}(s_0)$$

- If one takes as input α_s such that $A^{(0)}_{\mathrm{pert}}(s_0) \neq A^{(0)}(s_0)$, one needs to compensate with $\Delta A^{(0),\mathrm{DV}}(s_0)$
- But
  - Spectral function is already very close to the partonic prediction
  - Asymptotic freedom requires $\Delta A^{(0),\mathrm{DV}}(s_0) \to 0$ quite fast



Small α_s can only be obtained with artificial Heaviside-like shapes

Test standard assumptions with DV models: pinched

$$A^{\omega}(s_0) = A^{\omega}_{\mathrm{pert}}(s_0) + A^{OPE,D>0}(s_0)$$

- Perturbation theory alone with $\alpha_s^{\mathrm{FOPT}} \sim 0.32$ also matches all pinched-moments well
- If one takes as input α_s such that $A^{(0)}_{\mathrm{pert}}(s_0) \neq A^{(0)}(s_0)$, one needs to artificially compensate by tuning arbitrarily large $\mathcal{O}_D$ (add as many parameters as observables)

Weight	variation	Pert	$\mathcal{O}_{2(n+2),V+A}$	$\mathcal{O}_{2(n+3),V+A}$	DV	Exp
$A^{\omega\,(2,1)}_{V+A}(m_\tau^2)$	Default	0.0938 (5)	0.0029	-0.0019	-0.0001	**0.0954 (3)**
	1	0.0952 (7)	-0.0001	-0.0004	-0.0000	0.0954 (3)
	2	0.0957 (8)	-0.0010	0.0000	-0.0000	0.0954 (3)
	3	0.0908 (2)	0.0145	-0.0095	-0.0007	0.0954 (3)
	4	0.0958 (8)	-0.0011	-0.0005	-0.0000	0.0954 (3)
$A^{\omega\,(2,4)}_{V+A}(m_\tau^2)$	Default	0.1316 (4)	0.0025	-0.0007	0.0001	**0.1344 (8)**
	1	0.1331 (5)	0.0009	-0.0004	0.0001	0.1344 (8)
	2	0.1336 (5)	0.0004	-0.0001	0.0000	0.1344 (8)
	3	0.1282 (2)	0.0171	-0.0061	-0.0056	0.1344 (8)
	4	0.1337 (5)	-0.0002	0.0002	0.0001	0.1344 (8)

Test standard assumptions with DV models: pinched

$$A^{\omega}(s_0) = A^{\omega}_{\rm pert}(s_0) + A^{OPE,D>0}(s_0)$$

- Perturbation theory alone with $\alpha_s^{\rm FOPT} \sim 0.32$ also matches all pinched-moments well
- Their size becomes more than questionable when $\alpha_s < 0.3$ and would imply complete breakdown of the OPE at $\hat{s}_0$ (unjustified and inconsistent)

Weight	variation	Pert	$\mathcal{O}_{2(n+2),V+A}$	$\mathcal{O}_{2(n+3),V+A}$	DV	Exp
$A^{\omega(2,1)}_{V+A}(\hat{s}_0)$	Default	0.1010 (18)	0.0248	−0.0326	0.0062	0.0994 (4)
	1	0.1043 (28)	−0.0006	−0.0071	0.0028	0.0994 (4)
	2	0.1054 (32)	−0.0081	0.0003	0.0018	0.0994 (4)
	3	0.0948 (06)	0.1221	−0.1629	0.0452	0.0994 (4)
	4	0.1010 (18)	0.0042	0.0015	−0.0001	0.0980 (3)
$A^{\omega(2,4)}_{V+A}(\hat{s}_0)$	Default	0.1391 (10)	0.1808	−0.1012	−0.0787	0.1401 (5)
	1	0.1424 (14)	0.0676	−0.0572	−0.0128	0.1401 (5)
	2	0.1434 (16)	0.0281	−0.0203	−0.0112	0.1401 (5)
	3	0.1327 (05)	1.2216	−0.8833	−0.3309	0.1401 (5)
	4	0.1392 (11)	−0.0036	0.0058	−0.0034	0.1378 (4)

In view of the results, further inflation of uncertainties is not justified

Conclusions

- One of the most precise phenomenological determinations of α_s comes from inclusive tau decays
- Nonperturbative effects are tiny, but poorly known
- Current modelling of duality violations is subject to large systematic uncertainties
- Our final estimate gives

$$\alpha_s^{(n_f=3)}(m_\tau^2) = \begin{cases} 0.335 \pm 0.013 & \text{(CIPT)} \\ 0.320 \pm 0.012 & \text{(FOPT)} \end{cases}$$

or

$$\alpha_s^{(n_f=5)}(M_Z^2) = 0.1197 \pm 0.0015$$