QCD Spectral Sum Rules 2022
(QSSR 22)

Stephan Narison

Survivor of the 1979 Sum Rule Generation !

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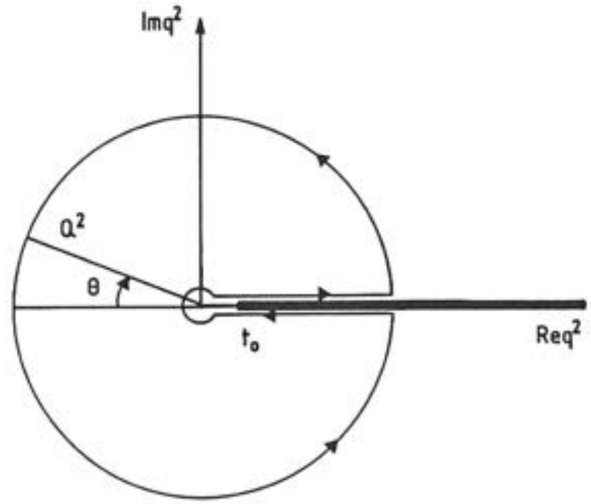

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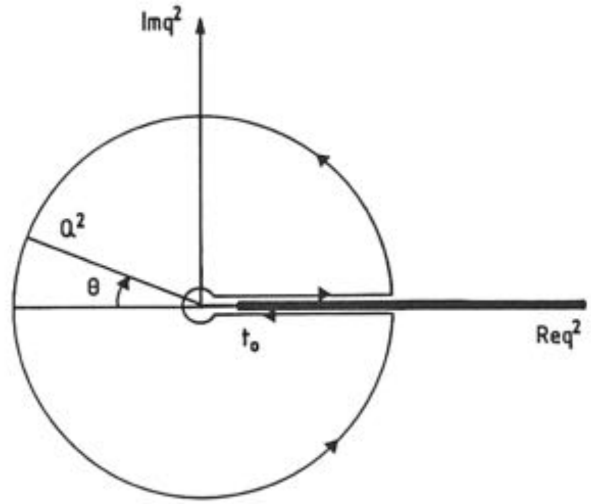

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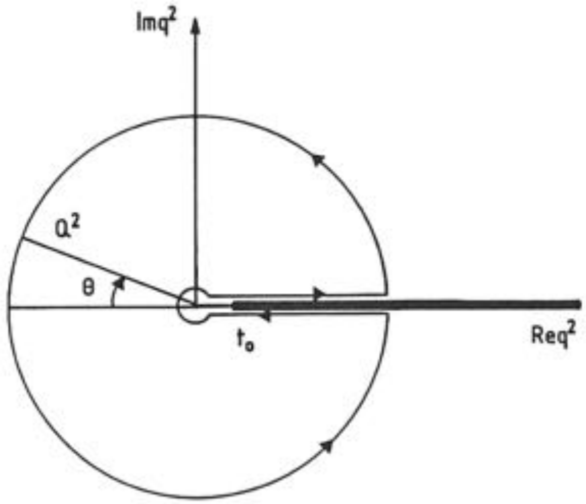

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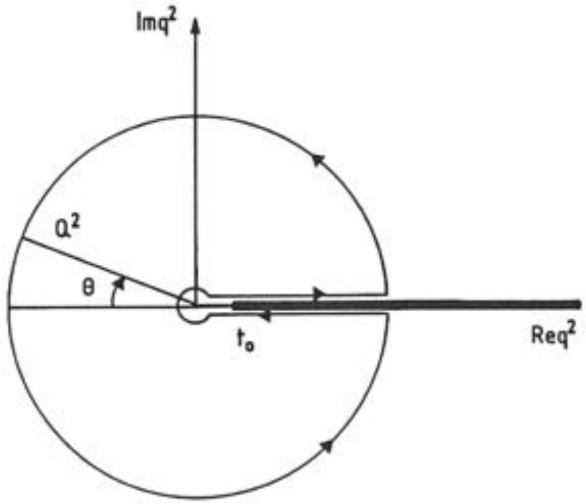

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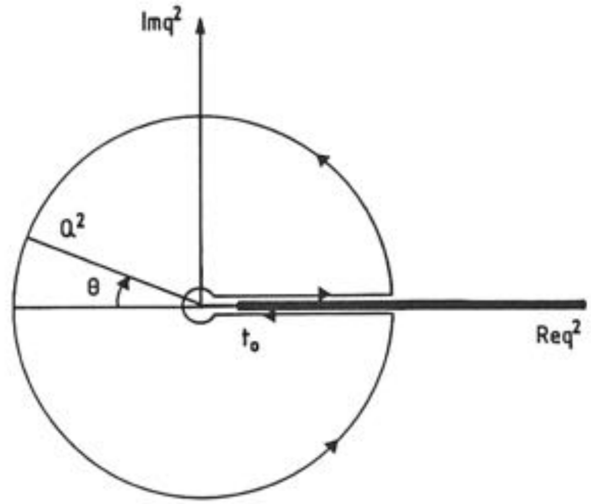

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- $\frac{1}{\pi}\mathrm{Im}\Pi(t) \sim \sigma_{tot}(e^+e^- \to \rho, J/\psi, \Upsilon, \ldots) : \bar\psi\Gamma\psi \ \ : \ \Gamma = \gamma_\mu$: Complete Data

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- Asympotic $SU(n)_L \otimes SU(n)_R$ chiral symmetry Weinberg 1967

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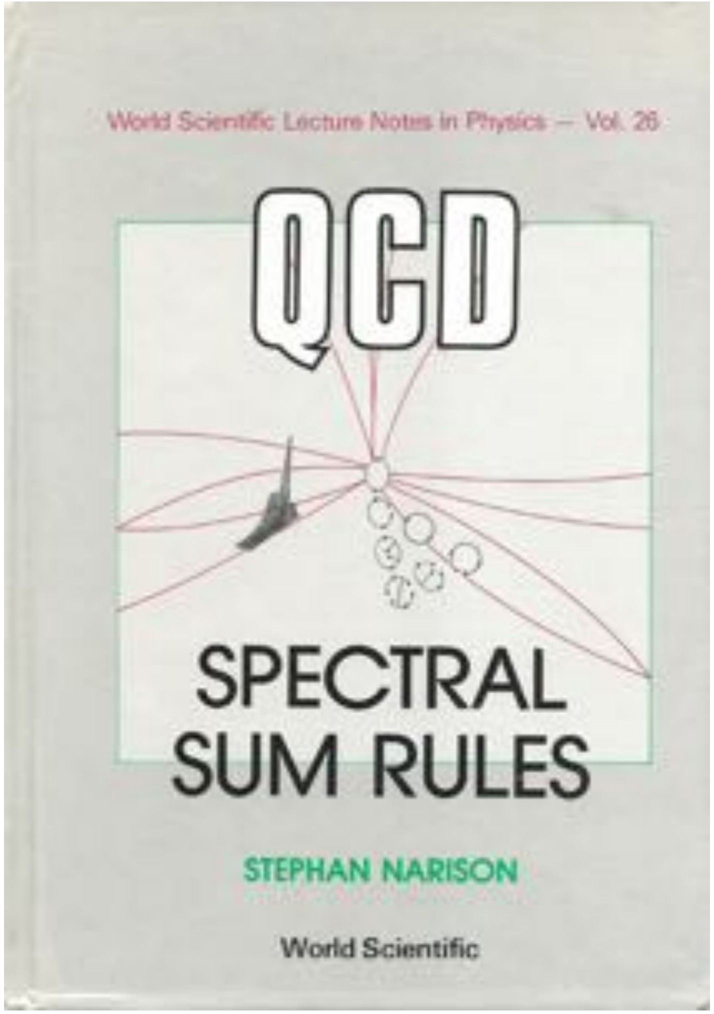
- Shifman Vainshtein Zakharov : Sakurai's price 1999

Photo : Munich 2006

Introductory Books and Reviews

1989



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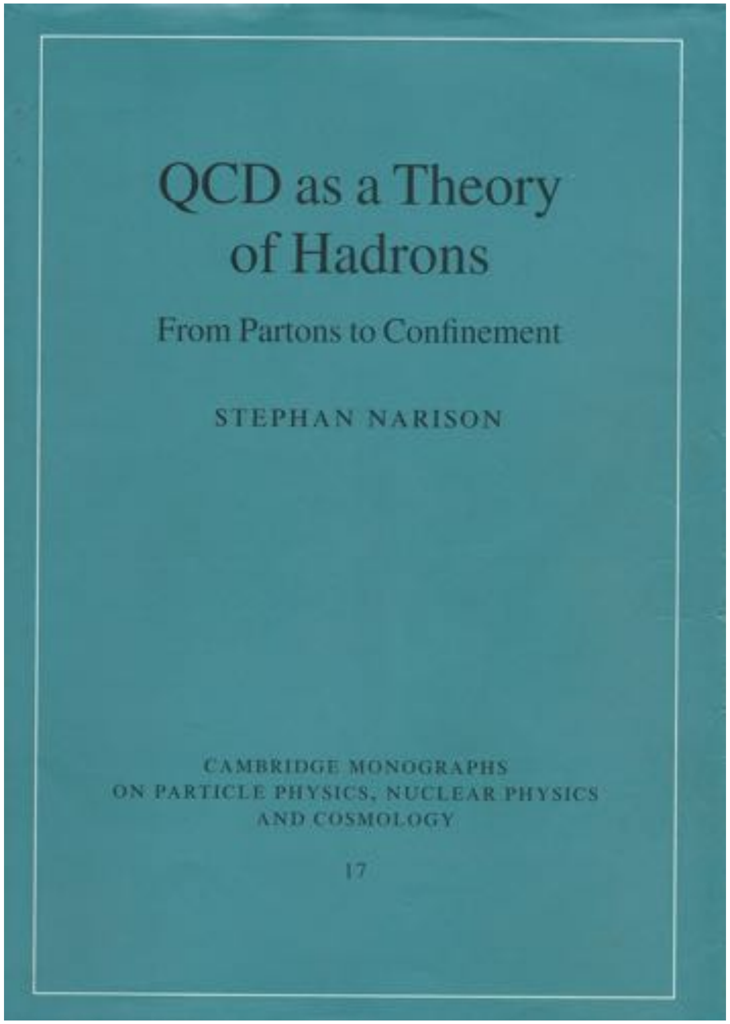

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  - Moment Sum Rules for Heavy Quarks SVZ 79

$$\mathcal{M}_n\left(Q^2\right) = \int_{4m_Q^2}^{\infty} dt \frac{\text{Im}\Pi(t)}{(t+Q^2)^{n+1}}, \quad r_{n/n+j}(Q^2) = \frac{\mathcal{M}_n(Q^2)}{\mathcal{M}_{n+j}(Q^2)} : \; n, j = 1, 2...$$

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  - No good control of condensates beyond $p = 3$: factorization, mixing under renormalization, often some classes of diagrams are only computed,...

Optimal Results from Stability Criteria

Principle of Minimum Sensitivity of the Obervables (M_G, f_G) versus the external variables (τ, t_c, μ)

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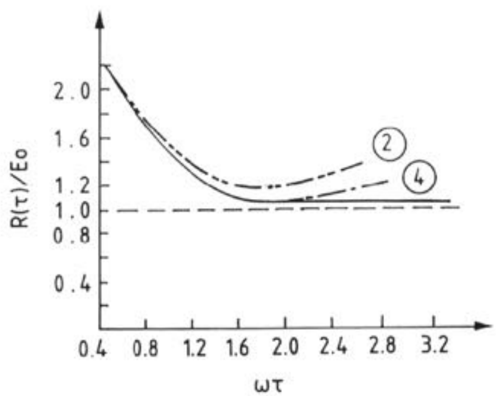
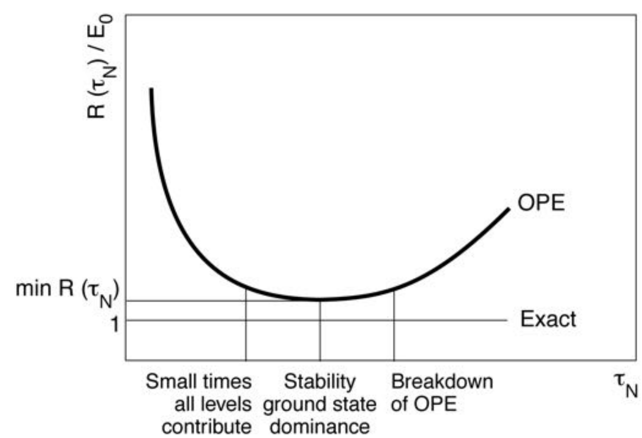
Non-Rel. J/ψ sum rules





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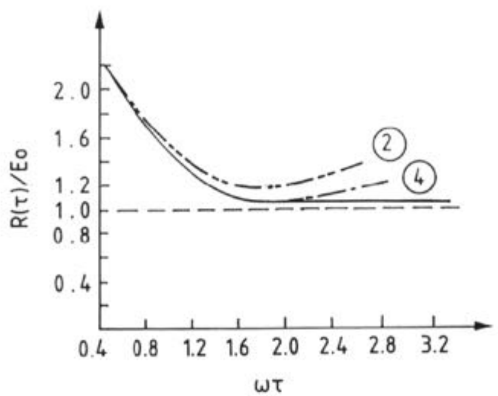

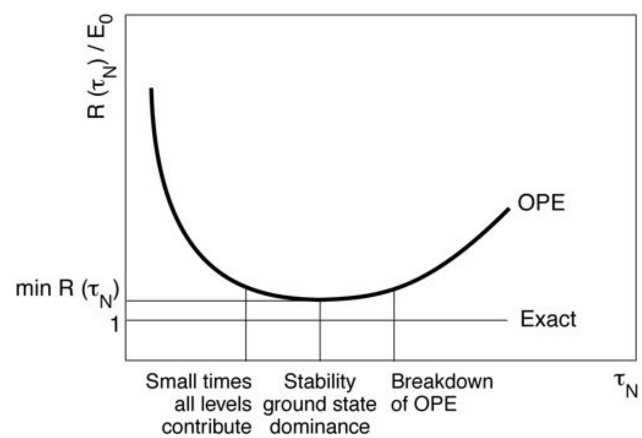

  - Checked in Different Channels

    SN books 1989 & 2002 and $\neq$ SN papers

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- Improvement using Finite Width Correction : 0^{++} Gluonium (σ) sum rules but... a little Gain S.N Veneziano 89

QCD parameters from Light Quarks SR

Observables	Estimate [Gev]^d	$\geq$ Bound	Source	Refs	PDG
Masses					
$\overline{m}_{ud}(2)$	3.95 ± 0.28	(3.28 ± 35)	$\pi \oplus \pi'$	SN15	$3.45^{+0.55}_{-0.17}$
$\overline{m}_u(2)$	2.64 ± 0.28	2.19 ± 0.27	–	"	$2.16^{+0.49}_{-0.26}$
$\overline{m}_d(2)$	5.27 ± 0.49	4.37 ± 0.54	–	"	$4.67^{+0.48}_{-0.17}$
$\overline{m}_s(2)$	98.5 ± 5.5	81.6 ± 4.5	$K \oplus K'$	"	93^{+11}_{-5}
$\overline{m}_s/\overline{m}_{ud}$	24.9 ± 2.3	–	$\pi \oplus K$	"	$27.3^{+0.7}_{-1.3}$
$\alpha_s\lambda^2$ tach. gluon	$-(265 \pm 57)^2$	–	π, e^+e^-	SN95,07, CNZ	
Condensates					
$\langle\bar{u}u\rangle$	$-(276 \pm 7)^3$	–	$\pi \oplus \pi'$, Baryon	SN15,DJN10	–
$\langle\bar{s}s\rangle/\langle\bar{u}u\rangle$	0.74 ± 0.03	–	$\Omega_c \oplus \Xi_{c,b}$	ANN10	–
			$\pi \oplus K \oplus a_0$	SN	
$g\langle\bar{u}Gu\rangle \equiv M_0^2\langle\bar{u}u\rangle$	$M_0^2 = 0.8 \pm 0.2$	–	Baryon	Dosch 84,DJN89, Ioffe 81	–
$\alpha_s\langle\bar{u}u\rangle^2$	$(5.8 \pm 1.8) \times 10^{-4}$	–	e^+e^-, Baryon, τ	LNT84, Dosch 84, SN95,09	–
$\langle\alpha_s G^2\rangle/\alpha_s\langle\bar{u}u\rangle^2$	(106 ± 12)	–	e^+e^- ratio SR	$\Longrightarrow \langle\alpha_s G^2\rangle = (6.2 \pm 2.0)$ SN95	–

Gluon condensate $\langle \alpha_s G^2 \rangle$ prior 2017

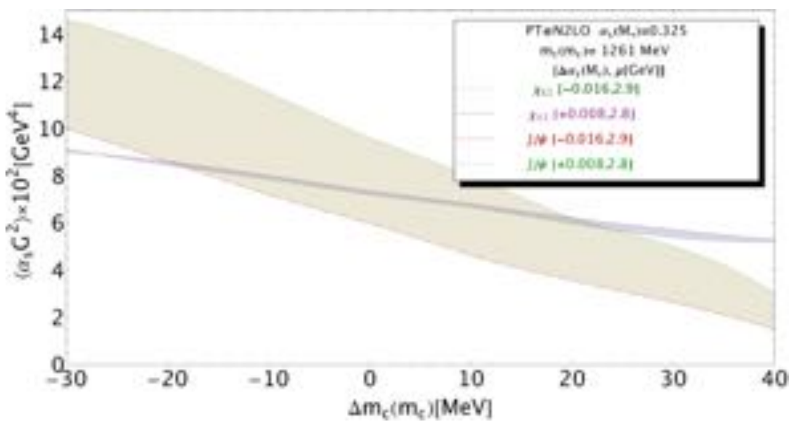
Sources	$\langle \alpha_s G^2 \rangle \times 10^2$ [GeV^4]	References
$e^+e^- \to$ I=1 Hadrons		
Exponential	$0.9 \sim 6.6^*$	Eidelman et al. 79
Ratio of Exponential	4 ± 1	Launer et al. 84
FESR	13 ± 6	Bertlmann et al. 88
Infinite norm	$1 \sim 30^*$	Causse-Mennessier
τ-like decay (ratio of LSR)	7 ± 1	SN 95
$\tau-$decay		
Axial spectral function	6.9 ± 2.6	Dominguez-Sola 88
Sum Rule Average	6.25 ± 0.45	**Prior 2017**
$\tau-$decay with high moments : good place for α_s... but not for $\langle \alpha_s G^2 \rangle$?		
ALEPH collaboration	6.3 ± 1.2	Duflot 95
CLEO II collaboration	2.4 ± 1.0	Duflot 95
OPAL collaboration	$-0.9 \sim +4$	Ackerstaff et al. 99
ALEPH collaboration	$-5 \sim +6$	Schael et al. 05
ALEPH collaboration	$-12 \sim -0.6$	Davier et al. 14
Lattice OK for the Order of Magnitude... but needs to be improved !		
O(α_s^{12}), O(α_s^{35}), $\langle$plaquette$\rangle$	$\approx 13,\ 27,\ 44$	Rakow 05, Bali-Pineda 15, Lee 14

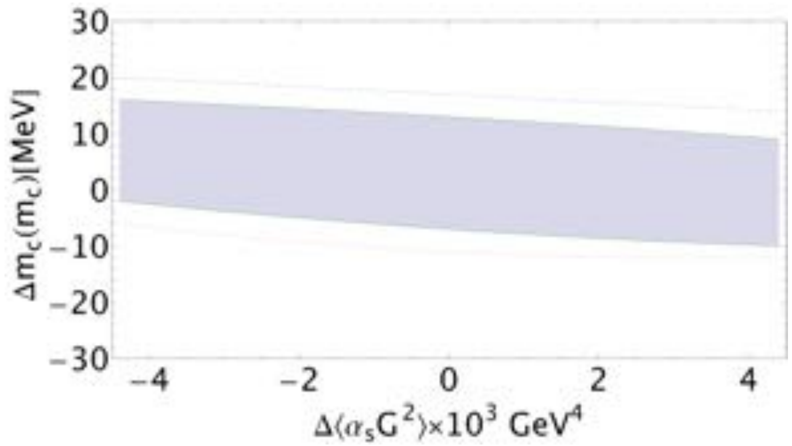
Gluon condensate $\langle \alpha_s G^2 \rangle$ prior 2017

Sources	$\langle \alpha_s G^2 \rangle \times 10^2$ [GeV^4]	References
Charmonium		
$q^2 = 0$-moments	4 ± 2	SVZ 79 (guessed error)
$q^2 \neq 0$-moments	5.3 ± 1.2	RRY 81-85
–	9.2 ± 3.4	Miller-Olssson 82
–	$\approx 6.6^*$	Broadhurst et al. 94
–	2.8 ± 2.2	Ioffe-Zyablyuk 07
–	7.0 ± 1.3	Narison 12a
Exponential	12 ± 2	Bell-Bertlmann-Neufeld 82
–	17.5 ± 4.5	Marrow et al. 87
–	7.5 ± 2.0	Narison 12b
Exponential $M_\psi - M_{\eta_c}$	10 ± 4	Narison 96
Bottomium		
Exponential $M_{\chi_b} - M_\Upsilon$	6.5 ± 2.5	Narison 96
Non-rel. moments	5.5 ± 3	Yndurain 99

m_c and $\langle \alpha_s G^2 \rangle$ Correlations *IJMP A33 (2018)n.10, 1850045*

- ♣ (Axial) Vector: $M_{\chi_{c1}}$, $M_{J/\psi}$ (Pseudo)Scalar : M_{η_c}





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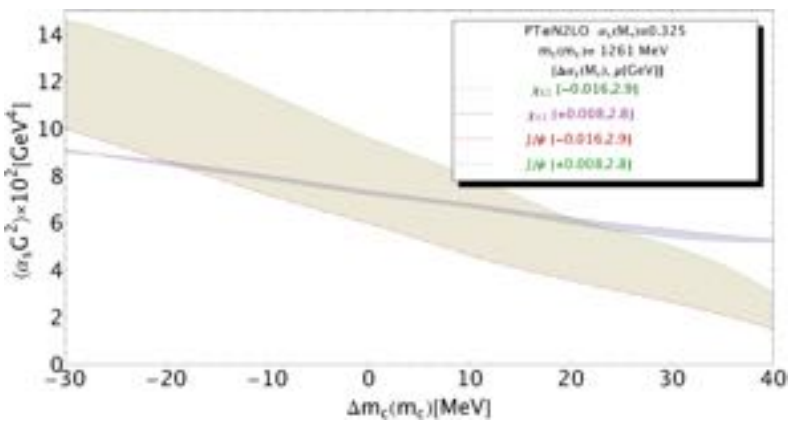

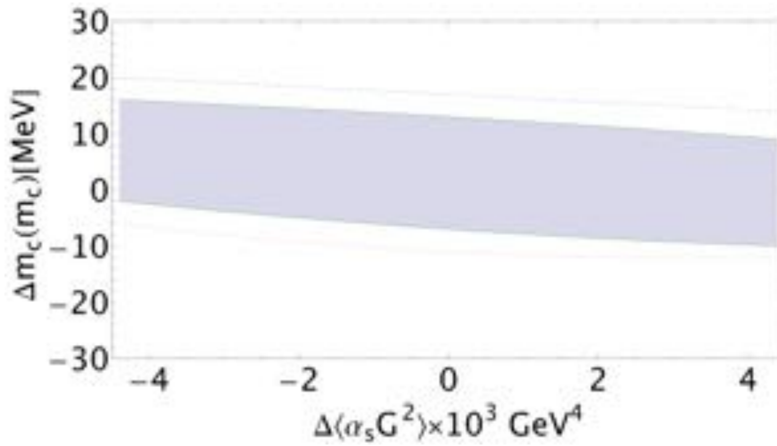

- ♢ Comments : J/ψ Moments [Ioffe 05, Ioffe-Zyablyuk 07]

  $\langle \alpha_s G^2 \rangle \approx 0.028$ GeV^4 $\Longrightarrow m_c \approx 1285$ MeV ...But for $m_c \approx 1265$ MeV $\Longrightarrow \langle \alpha_s G^2 \rangle \approx 0.062$ GeV^4

  J/ψ alone cannot fix accurately $\langle \alpha_s G^2 \rangle$.

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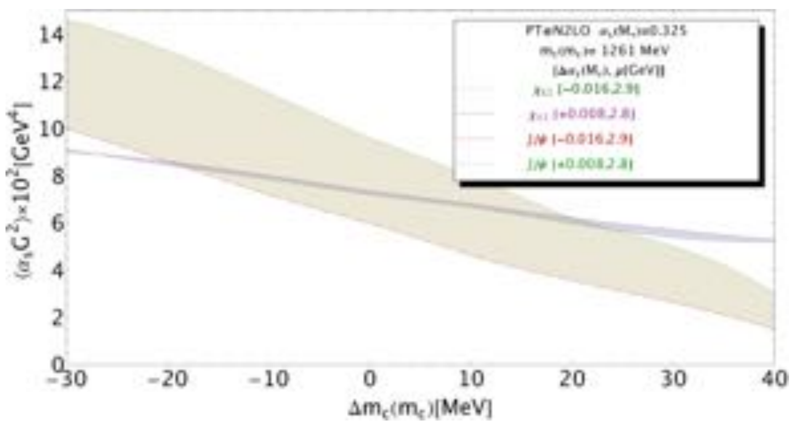

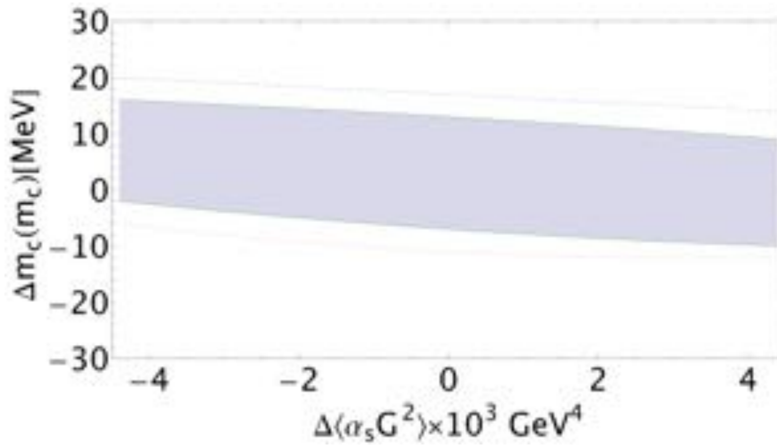

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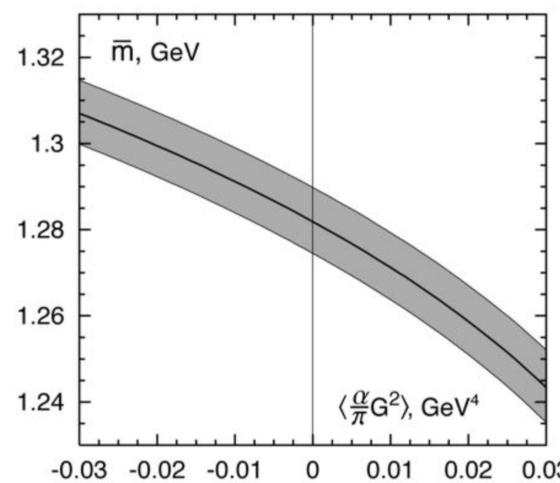
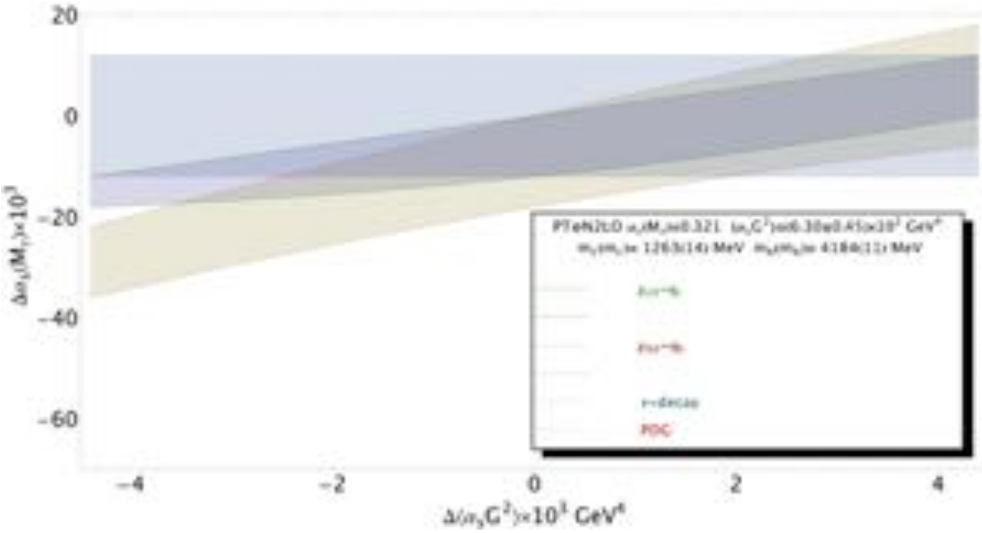
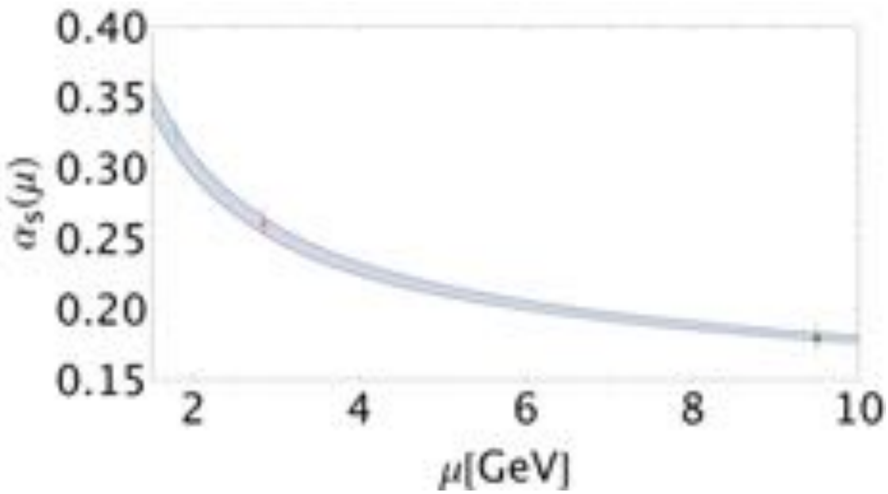


α_s *and* $\langle \alpha_s G^2 \rangle$ *Correlations* *IJMP A33 (2018)n.10, 1850045*

- ♣ Mass-splittings $M_{\chi_{c,b}} - M_{\eta_{c,b}}$





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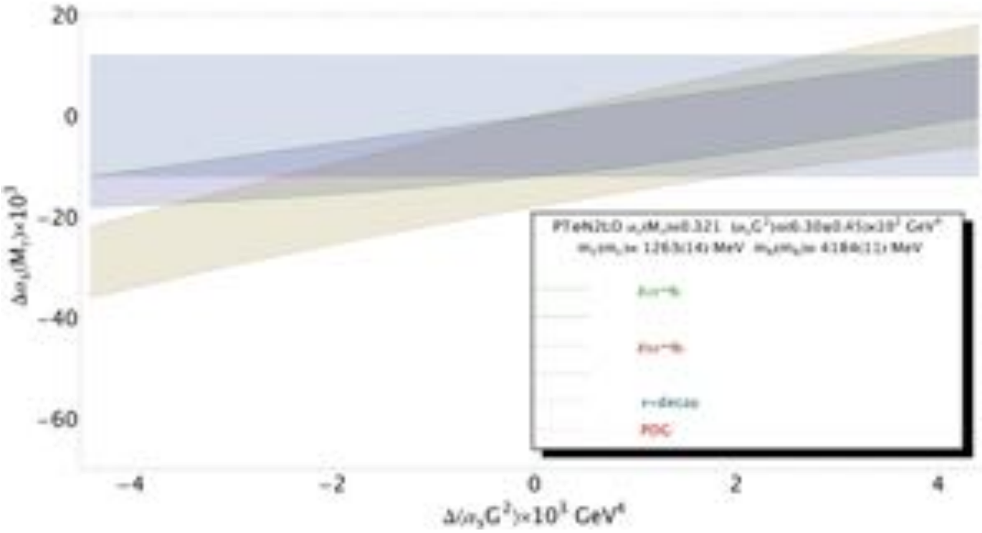

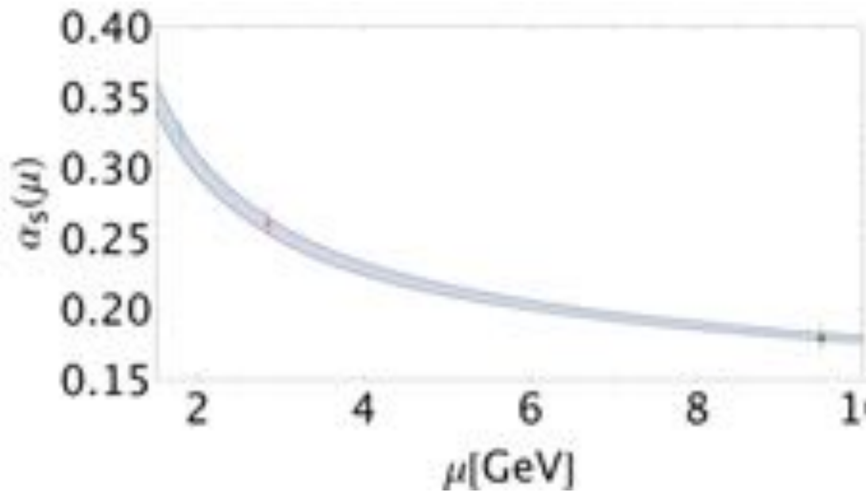

- ♢ Values of $\alpha_s(\mu)$ to N2LO

Inputs : $m_{c,b}(\mu)$, $\langle \alpha_s G^2 \rangle \Longrightarrow$

Charm : $\alpha_s(2.9) = 0.261(10)$, Bottom : $\alpha_s(9.0) = 0.1841(46)$

Mean $\Longrightarrow \alpha_s(M_Z) = 0.1181(16)(3)$

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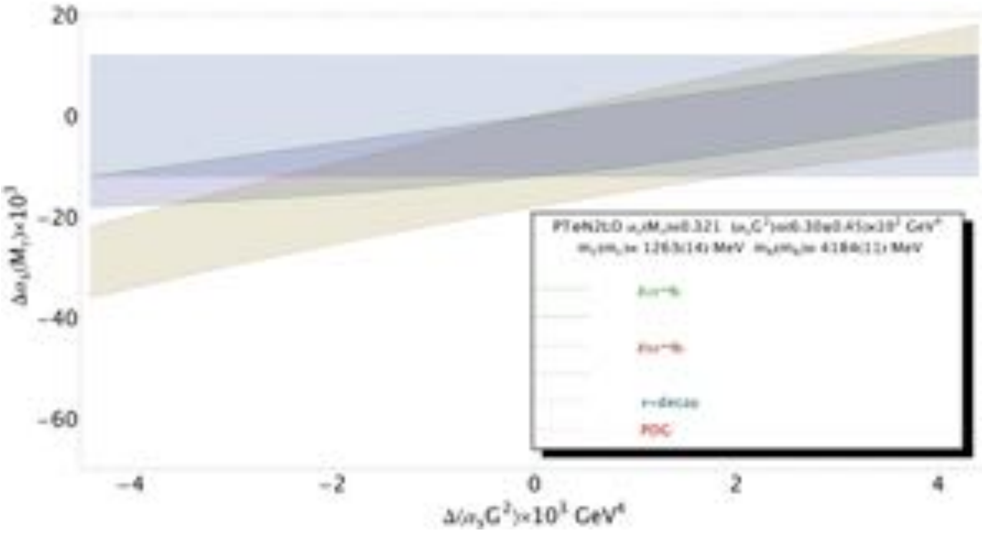

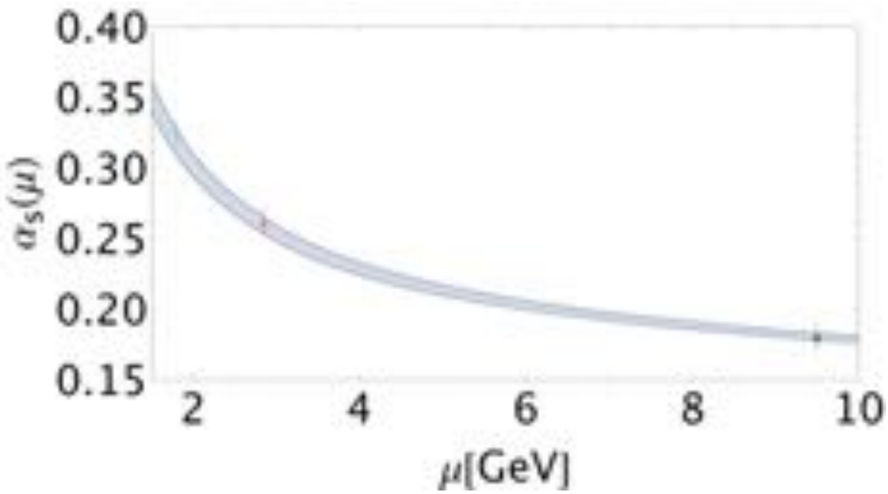

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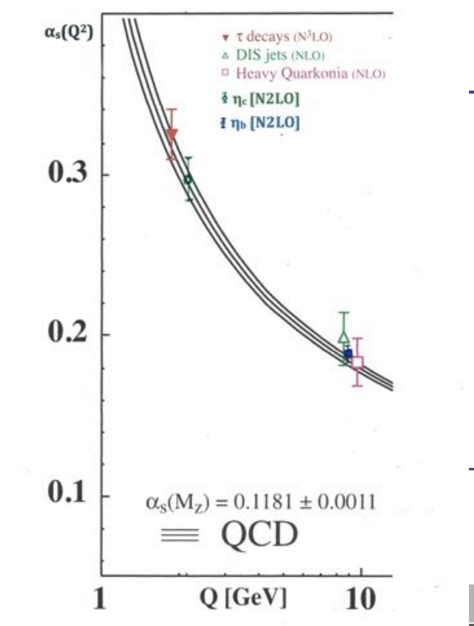
- ♡ For some other correlations see : SN IJMP A33 (2018)n.10, 1850045

QCD parameters from Heavy Quarks SR

Observables	Laplace SR	Moments	Mean	PDG
$\overline{m}_c(\overline{m}_c)$[MeV]	1265(13), 1275(11)	1264(6)	1266(6)	1270(20)
Sources	$M_{\chi_{c1}} \oplus J/\psi,\ M_{\eta_c},\ \ M_{B_c}$	J/ψ		
$\overline{m}_b(\overline{m}_b)$[MeV]	4192(17) 4216(10)	4188(8)	4199(8)	4180(30)
Sources	$\Upsilon,\ \ M_{B_c}$	Υ		
$\langle\alpha_s G^2\rangle$[GeV]$^4 \times 10^2$	(6.39 ± 0.35)	–	(6.35 ± 0.35)	SR Average
Sources	$M_{\chi_{c1}} \oplus J/\psi,\ M_{\chi_{c0,b0}} - M_{\eta_{c,b}}$			
$\langle g^3G^3\rangle/\langle\alpha_s G^2\rangle$[GeV]^2	(8.2 ± 1.0)	(8.8 ± 5.5)	(8.2 ± 1.0)	SN12
Sources	J/ψ	J/ψ		
α_s	$\alpha_s(2.9)$=0.261(10)			
	$\alpha_s(9)$=0.1841(46)			
Mean	$\Longrightarrow \alpha_s(M_Z) = 0.1181(16)(3)$			
Sources	$M_{\chi_{c0,b0}} - M_{\eta_{c,b}}$			



Heavy-light decay constants [MeV]

SN15 (Int.J.Mod.Phys.A 30 (2015) 20, 1550); Nucl.Part.Phys.Proc. 258-259 (2015) 189

0^-	D	D_s	D_s/D	B	B_s	B_s/B
LSR	204(5)	243(5)	1.170(23)	204(5)	235(4)	1.154(21)
LATT	212(1)	249(1)	1.173(3)	188(3)	227(2)	1.213(7)
DATA	204(7)	258(4)	–	196(24)	–	–
1^-	D^*	D_s^*	D_s^*/D^*	B^*	B_s^*	B_s^*/B^*
LSR	250(8)	290(11)	1.16(4)	210(6)	221(7)	1.064(10)
LATT	253(7)	283(5)	1.17(3)	188(3)	217(5)	1.064(11)
0^+	D_0^*	$D_{0_s}^*$	$D_{0_s}^*/D_0^*$	B_0^*	$B_{0_s}^*$	$B_{0_s}^*/B_0^*$
LSR	220(11)	202(15)	0.922(15)	278(12)	255(15)	0.865(54)
1^+	D_1	D_1/D	B_1	B_1/B		
LSR	363(11)	1.78(1)	385(18)	1.87(6)		

B_c-like : LSR $\oplus$ Heavy Quark Symmetry (HQS)

SN15 & Phys.Lett.B 807 (2020) 135522

- **Spectra** (HQS = Flavour Independence of Hyperfine splittings and Excitation Energy)

Channel	LSR	HQS	Lattice [Mathur...18]	PM [Quigg-Rosner 19]
$B_c^*(1^{--})$	$6451(86)$	6315(1)	6331(7)	6330(20) [Bagan et al. 94]
$B_{0c}^*(0^{++})$	6689(198)	6723(29)	6712(19)	6693
$B_1(1^{++})$	6794(128)	6730(8)	6736(18)	6731
$B_{c2}(2^{++})$	–	6741(8)	–	7007
$B_0^*(0^{++})$	5701(196) [SN05]	5733		

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- Decay constants

	$B_c(0^{--})$	$B_c^*(1^{--})$	$B_{0c}^*(0^{++})$	$B_{1c}(1^{++})$	$B_0^*(0^{++})$
LSR	371(17) [SN20]	442(44)	155(17)	274(23)	–
HQS	–	387(15)	158(9)	266(14)	271(26) [SN15]

Digluonia Couplings and Masses

Correlator : $\psi_{\mp}(q^2) \equiv i\int d^4x\, e^{-qx}\langle 0|J_{\mp}(x)\,(J_{\mp})^{\dagger}(0)|0\rangle$ with:

$J_{-}(x) \equiv (8\pi)Q(x) = (\alpha_s)\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\mu\nu,a}G^{\rho\sigma_a}$ $\qquad J_{+}(x) \equiv 4\theta^{\mu}_{\mu}(x) = \beta(\alpha_s)G^a_{\mu\nu}G^{\mu\nu}_a$

0^{-+}	f_G [MeV]	Mass [MeV]				
		QSSR		Models	Lattice	Data
η_1	905(72)	$825(45)^*$				$\eta(958)$
P_{1a}	594(144)	1338(112)				$\eta(1295)$
P_{1b}	594(144)	1462(117)		1400		$\eta(1405)$
P_1'	205(282)	1541(118)		1750		$\eta(1495, 1760)$
P_2	500(43)	2751(140)			2150-2720	
0^{++}	f_G [MeV]	Mass [MeV]	Data		Width [MeV]	Data
σ_B	456(157)	1070(126)	$f_0(0.5, 1.37)$		$\pi\pi$ 873	700 $\pi\pi$ scattering
σ_B'	329(30)	1121(117)	$f_0(0.5, 1.37)$		$2(\pi\pi)_S$ 186(35)	
G_1	365(110)	1548(121)	$f_0(1.5, 1.7)$		$\eta\eta'$ (2.5 ± 1.4)	(2.6 ± 0.9)
					$\eta\eta/\eta\eta'$ (2.3 ± 0.6)	3.0
G_1'	1000(230)	15631(141)	$f_0(1.5, 1.7)$			
G_2	797(74)	2992(221)	$f_0(2.02 - 2.2)$			

Conformal &Topological Charges / Proton Spin

- Conformal Charge and its Slope

  $\psi_+(0)|_{YM} = (2.09 \pm 0.29)\ \mathrm{GeV}^4 \qquad \psi'_+(0) = (0.95 \pm 0.30)\ \mathrm{GeV}^2$

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- Slope of the Topological Charge and the Prton Spin NSV revisited

$$\sqrt{\chi'(0)} \equiv \frac{1}{(8\pi)}\sqrt{\psi_-(0)}(Q^2 = 10\ \mathrm{GeV}^2) = (22.1 \pm 3.1)\ \mathrm{MeV}$$

$$\ll f_\pi/\sqrt{6} = 38\ \mathrm{MeV}\ (\mathrm{OZI})$$

$$\Longrightarrow G_A^{(0)}(\mathrm{GeV}^2) = (0.337 \pm 0.050) \quad [\mathrm{Exp}\ : 0.35 \pm 0.12]$$

$G_A^{(0)}(\bar{u}\gamma_5 u) = \frac{1}{2M_p}(2n_f)\sqrt{\chi'(0)}\Gamma_{\Phi_{5R}\bar{P}P}$: SV

Φ_{5R} : renorm. bilinear quark current ; $\Gamma_{\Phi_{5R}\bar{P}P}$ RG invariant proper vertex

For OZI :

$\Gamma_{\Phi_{5R}\bar{P}P}|_{\mathrm{OZI}} = \sqrt{2}g_{\eta 8PP}(\bar{u}\gamma_5 u), \sqrt{\chi'(0)}|_{\mathrm{OZI}} = f_\pi/\sqrt{6} = 38\ \mathrm{MeV}, G_A^{(0)}|_{\mathrm{OZI}} = 0.579 \pm 0.021.$

Some Other Exotics

- Four-quark / Molecule States

  See the talks of R. Albuquerque, M. Nielsen, D. Rabetiarivony

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  - QSSR remains competitive (results obtained many years before) compared with Lattice $\cdots$ if done carefully !