The Taylor–Lagrange scheme as a template for symmetry-preserving renormalization procedures

2011 J. Phys. A: Math. Theor. 44 385402

(http://iopscience.iop.org/1751-8121/44/38/385402)

J. Phys. A: Math. Theor. **44** (2011) 385402 (26pp) doi:10.1088/1751-8113/44/38/385402

The Taylor–Lagrange scheme as a template for symmetry-preserving renormalization procedures

P Grangé[1] **and E Werner**[2]

[1] Laboratoire Univers et Particules, Université Montpellier II, CNRS/IN2P3, Place E. Bataillon, F-34095 Montpellier Cedex 05, France
[2] Institut für Theoretische Physik, Universität Regensburg, Universitätstrasse 31, D-93053 Regensburg, Germany

E-mail: grange@lupm.univ-montp2.fr and Ernst.Werner@physik.uni-regensburg.de

Received 12 September 2010, in final form 11 July 2011
Published 5 September 2011
Online at stacks.iop.org/JPhysA/44/385402

Abstract
A general regularization/renormalization scheme based on intrinsic properties of quantum fields as operator-valued distributions with adequate test functions is presented. The paracompactness property of the Minkowskian or Euclidean manifolds permits a unique definition of fields through integrals over the manifold based on test functions which are partition of unity (PU). These test functions turn out to provide a direct Lorentz-invariant scheme to the extension procedure of singular distributions and possess the unique property of being equal to their Taylor remainder of any order. When expressed through Lagrange's formulae, this remainder leads to specific procedures of extension in the UV and IR domains. These results, directly obtainable at the physical dimension $D = 4$, are found to depend on an arbitrary scale present in the definition of any PU test functions and relevant to the final RG-analysis of physical amplitudes. The method is of general character in that it comprises the well-known symmetry-preserving procedures of Bogoliubov–Parasiuk–Hepp–Zimmermann, Pauli–Villars subtractions at the level of propagators and dispersion relations. Symmetry preservation is indeed verified explicitly in simple QED/QCD gauge-boson contributions and in a covariant light-front dynamics treatment of the Yukawa model.

PACS numbers: 11.10.Ef, 11.10.Gh, 11.10.−z, 03.07+k

1. Introduction

In quantum field theory (QFT), most known regularization and renormalization procedures for perturbative calculations of physical observables rely on identification and cancellation of divergences via appropriate counterterms. An exception is the Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) [1, 2] method which always yields converging Feynman integrals without any intermediate regularization and is therefore more satisfying from a mathematical point of view; however, it is rarely used because it is not the easiest one for practical calculations.

While the regularization by counterterms can be done in principle to any order in perturbation theory, the procedure becomes difficult and intricate beyond leading orders. A very convenient way of dealing with infinities in gauge theories is *dimensional regularization* [3] which preserves gauge invariance. However, for supersymmetric theories, it has problems because invariance under supersymmetric transformations holds only for entire spacetime dimensions. To circumvent this difficulty, *dimensional reduction* was proposed [4] where the field components are unchanged in order to preserve supersymmetry. However, ambiguities arise related to the treatment of the Lévi-Cività symbol $\epsilon^{\mu\nu\rho\sigma}$ and of γ^5. In addition, there can be complications with unitarity due to *evanescent couplings* when dimensional reduction is applied to non-supersymmetric theories as e.g. the standard model.

From the late nineties until now a number of important papers have appeared dealing with these problematics occuring throughout the very foundations of QFT. Some of the most recent works we shall refer to in the sequel are mainly concerned with renormalization issues and the aim to build finite field theories. Thus renormalization does not really appear as an outdated subject from the seventies, as sometimes stated, and the search is still open [5] for other regularization schemes which do not suffer from the above-listed problems.

In this paper, we present a very general regularization/renormalization scheme based on the definition of quantum field operators as operator-valued distributions (OPVDs). The foundations of this approach go back to the early QFT history. A general mathematical framework was presented in the seventies [6] and revised in the late nineties [7]. The test functions necessary to smear out the field operators in coordinate space present at first sight a high degree of arbitrariness seemingly fatal to the uniqueness of the resulting field theory. Moreover, it is a common lore that the cut-off procedures in Fourier space associated with the smearing in coordinate space eventually spoil the symmetry properties of the theory.

Our aim is to show the following propositions.

(A) (i) the uniqueness requirement—that is the independence with respect to the form of the test functions used for the definition of the physical fields—and (ii) Lorentz and Poincaré invariances ensue from the well-established nature of Minkowskian and Euclidean spaces as paracompact entities equipped with test functions as partitions of unity (PU) [8, 9];

(B) the PU test functions can be set up in such a way (running support, ultrasoft cut-off) that the above-mentioned symmetry violations can be avoided;

(C) a very effective way to handle singular distributions is via the extension of distributions, a well-established method in the mathematical literature [10, 11], advocated in QFT a while ago by Prange [12], and Brunetti and Fredenhagen [13];

(D) the proposed method allows us to treat simultaneously on an equal footing ultra-violet and infrared singularities;

(E) equivalent mathematical transcriptions in the ultraviolet relate our treatment to the known procedures commonly used in the past, hence our present title for this work.

The central idea to reach these goals is to transfer the regularizing effect of test functions in functionals onto modified but regular distributions constructed from the original singular ones. Therefrom, the resulting QFT is free of divergences and renormalization is always finite. By the very nature of the PU definition, the extended distributions contain a scale relevant to renormalization group studies. In a rather short recent account of the OPVD approach [14], we called the method Taylor–Lagrange renormalization scheme (TLRS) because technically the starting point in the extension of singular distributions is a Taylor surgery combined with Lagrange's integral formula for the Taylor remainder.

The paper is organized as follows. In section 2, we present first the definition of classical Euclidean fields within the continuous linear functional approach, the choice of the topological vector space of test functions and the occurence of partitions of unity as eligible test functions of rapid decrease in the sense of L. Schwartz. Then the transition to OPVD's and their Hilbert space properties is made explicitely for the free scalar field of positive mass in a 4-dimensional Minkowski flat space $\mathcal{M}$. In section 3, a short survey on the extension of singular distributions is given and TLRS is shown to encompass BPHZ renormalization as a special case. Section 4 introduces the concept of *running support* of the PU test function in order to go beyond the usual regularization method with a sharp *cut-off* while preserving symmetries. The mechanisms induced by the specific properties of a PU super regular test function (SRTF) are then detailed for the ultraviolet and infrared extensions of singular distributions. In the last section, we focus on the procedure of causal splitting of distributions with test functions and the ensuing link of the TLR approach with dispersion relations. Technical points and simple examples are treated in appendices.

2. Fields as OPVD–Lorentz invariance–Poincaré algebra

2.1. Distributions and test functions

In classical electrostatics in $\mathbb{R}^3$, the Coulomb field has components $E^i = \frac{q}{4\pi\epsilon_0}.\frac{x^i}{r^3}$ with $r^2 = \Sigma_{i=1}^3 (x^i)^2$, which are highly singular at $r = 0$. Old-timer pragmatic physicists *regularized* the field making the substitution $r \curvearrowright \sqrt{r^2+\epsilon^2}$. The resulting charge distribution $\rho(\epsilon) = \epsilon_0 \nabla \cdot \mathbf{E}$ is now smooth throughout the complete Euclidean space, with a total charge $Q(\epsilon) = 4\pi \int_0^\infty \rho(\epsilon) r^2 \, \mathrm{d}r = q$, independent of ϵ. The final result involves the *generalized δ-function* or *distribution* introduced by A M Dirac as $\lim_{\epsilon\to 0}\rho(\epsilon) = q\delta^{(3)}(\mathbf{x})$. It is a general feature that fields are not regular functions in the usual sense but distributions.

The theory of distributions we shall follow here is the Schwartz–Sobolev functional approach [8, 15, 16]. It is well documented in many text books [17] and we shall only give some basic ideas and properties necessary for our purposes. In the functional approach, a distribution T is not studied at its point values but as a continuous linear functional $T[\rho] \equiv \langle T, \rho\rangle$ of a *test function* $\rho \in \Xi(\mathbb{R}^n)$, the topological vector space of test functions. For instance, $\delta^{(3)}[\rho] = \rho(\mathbf{0})$. The distributions corresponding to Ξ are elements of the dual vector space Ξ'. The selection of the appropriate space Ξ depends on the problem under consideration. On a general ground, the stronger the topology of Ξ, the weaker that of Ξ' will be.

2.2. Classical Euclidean fields as distributions

We begin this subsection with a number of notations, definitions and theorems useful for the logical developments to follow.

Definition 1. *Let Ω be an open subset of $\mathbb{R}^n$. The genuine topological vector space $\mathcal{D}(\Omega)$ of [8] is the space of $\mathcal{C}^\infty$ test functions ρ such that supp $\rho \subset \Omega$.*

To deal with Fourier transforms it is necessary to enlarge the topological space $\mathcal{D}(\doteq \mathcal{D}(\mathbb{R}^n)$ and similar notations in other cases) of test functions with compact support to the space $\mathcal{S}$ of $\mathcal{C}^\infty$ test functions ρ of rapid decrease.

Notation 1. Let $\alpha = \{\alpha_1, \ldots, \alpha_n\} \in \mathbb{N}^n$ be a multi-index; we set $|\alpha| = \Sigma_{i=1}^n \alpha_i$, $\alpha! = \Pi_{i=1}^n \alpha_i!$, $x^\alpha = \Pi_{i=1}^n x_i^{\alpha_i}$ and

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \ldots \partial {x_n}^{\alpha_n}}$$

is a partial differential operator of order $|\alpha|$.

Definition 2. *For* $\mathbf{x} \in \mathbb{R}^n \quad (r^2 = \Sigma_{i=1}^n (x^i)^2)$ *a function* ρ *is of rapid decrease when it verifies*

$$\lim_{r\to\infty} |x^\alpha \partial^\beta \rho(\mathbf{x})| \to \mathbf{0} \quad \forall\{\alpha, \beta\} \in \mathbf{N^n}.$$

Test functions of compact support are a sub-class of functions of rapid decrease as easily seen on simple examples, e.g. like

$$\rho : x \mapsto \rho(x) = \begin{cases} \exp\left(\dfrac{1}{(x-a)(x-b)}\right) & x \in \,]a, b[\\ 0 \quad x \in \,]-\infty, a[\cup]b, +\infty[\end{cases}$$

and $\rho : x \mapsto \rho(x) = \exp(-x^2) \quad x \in \mathbb{R}$. *Then* $\mathcal{D} \subset \mathcal{S}$. *On the other hand the dual space of* $\mathcal{S}$, $\mathcal{S}'$, *is a subspace of* $\mathcal{D}'$: $\mathcal{S}' \subset \mathcal{D}'$. $\mathcal{S}'$ *is the space of the so-called tempered distributions.*

Tempered distributions are of central interest in many areas of physics, in particular in any QFT formulation. Many text books do collect the essential knowledge on tempered distributions and we shall not dwell on them further. Unless otherwise stated all proofs of properties and theorems invoked in the sequel can be found for instance in [8, 10, 16].

The topology of the space $\mathcal{S}(\mathbb{R}^n)$ is defined by a countable set of norms $\mathcal{N}_p$, $p \in \mathbb{N}$, such that

$$\mathcal{N}_p(\rho) = \sum_{|\alpha|,|\beta|\leqslant p} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \rho(x)|, \qquad p \in \mathbb{N}.$$

Convergence in $\mathcal{S}(\mathbb{R}^n)$ is defined consequently,

Definition 3. *A sequence of functions* $(\rho_j)_{j\geqslant 1} \in \mathcal{S}(\mathbb{R}^n)$ *converges to a function* $\rho \in \mathcal{S}(\mathbb{R}^n)$ *if* $\mathcal{N}_p(\rho_j - \rho) \to 0$ *for all* $p \geqslant 0$ *when* $j \to \infty$.

The continuity of the tempered distribution T *at* ρ_1 *states that for any* $\epsilon > 0$ *there exists an integer* p *and a* $\delta > 0$ *such that* $\mathcal{N}_p(\rho_2 - \rho_1) < \delta$ *implies* $|T(\rho_2) - T(\rho_1)| < \epsilon$.

Following Sobolev, and for later purposes, it is useful to consider further enlargements of $\mathcal{S}(\mathbb{R}^n)$ which are naturally Hilbert spaces [10, 15, 18].

Definition 4. *For* Ω *an open subset of* $\mathbb{R}^n$ *and* $s \in \mathbb{N}$ *one defines*

$$H^s(\Omega) = \{\rho \in L^2(\Omega) | \partial^\alpha \rho \in L^2(\Omega) \textit{ for } |\alpha| \leqslant s\},$$

with the Hilbert scalar product

$$(\rho|\sigma)_{H^s(\Omega)} = \sum_{|\alpha|\leqslant s} (\partial^\alpha \rho | \partial^\alpha \sigma)_{L^2(\Omega)} \quad \textit{with} \quad (f|g)_{L^2(\Omega)} = \int_\Omega \overline{f(x)} g(x)\, \mathrm{d}x.$$

Definition 5. *The Fourier transform* $\mathcal{F}[T]$ *of a tempered distribution* T *is defined by its action on the test function* $\rho \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \mathcal{F}[T], \rho \rangle \stackrel{\text{def}}{=} \langle T, \mathcal{F}[\rho] \rangle$$

This transform on the test function (resp. tempered distribution) corresponds to an isomorphism of $\mathcal{S}(\mathbb{R}^n)$ (resp. $\mathcal{S}'(\mathbb{R}^n)$) to itself.

Notation 2. We shall denote $\mathcal{F}[T]$ as $\tilde{T}$ and $\mathcal{F}[\rho]$ as $\tilde{\rho}$.

Remark 1. For a smooth flat manifold covered by a single coordinate system it is a general result that if $\rho(y) = \rho(-y)$ then $\tilde{\rho}(p) \equiv f(p^2)$ (e.g. in $\mathbb{R}^3$ $\mathcal{F}[\frac{1}{r}] = \frac{4\pi}{p^2}$, $p^2 = \Sigma_{i=1}^3 (p_i)^2$). Therefore a reflection symmetric $\rho(x-y) = \rho(y-x)$ has a well defined Fourier decomposition

$$\rho(x-y) = \int_{\mathbb{R}^n} \frac{\mathrm{d}^n q}{(2\pi)^n} \mathrm{e}^{\mathrm{i} \ll q, x-y \gg} f(q^2),$$

where $\ll q, x \gg$ is the bilinear form associated to the metric on $\mathbb{R}^n$ (cf below).

Remark 2. When the open subset Ω is the entire space $\mathbb{R}^n$ essential properties of $H^s(\mathbb{R}^n)$ can be established conveniently via Fourier transforms. In particular Sobolev spaces for non integer values of s are specified by

$$\text{for all} \quad s \in \mathbb{R} \quad H^s(\mathbb{R}^n) = \{T \in \mathcal{S}'(\mathbb{R}^n) | (1+p^2)^{s/2} \tilde{T} \in L^2(\mathbb{R}^n)\}.$$

This definition and the ensuing norm $\|.\|_{H^s(\mathbb{R}^n)}$ show (lemma 5.7.2 and proposition 5.7.4 of [19]) that the linear application

$$T : H^s(\mathbb{R}^n) \ni \rho \mapsto T[\rho] = \mathcal{F}^{-1}((1+p^2)^s \tilde{T})[\rho] \in H^{-s}(\mathbb{R}^n)$$

is an isometric isomorphism.

Finally the following overall embedding takes place

$$\mathcal{D} \subset \mathcal{S} \subset H^s \subset \mathrm{L}^2 \subset H^{-s} \subset \mathcal{S}' \subset \mathcal{D}'.$$

We consider now some specific properties of tempered distributions.

Theorem 1 (Chapter VII, theorem VI of [8]). *All tempered distributions T of $\mathcal{S}'(\mathbb{R}^n)$ can be expressed as $T = \partial_x^\alpha((1+r^2)^p g)$, where $\alpha \in \mathbb{N}^n$, p is a natural integer, g is a continuous bounded function on $\mathbb{R}^n$ and the partial derivative is in the sense of distributions.*

Remark 3. If a function in the usual sense is tempered its derivative may not be tempered: e.g. $\sin(e^x)$ is bounded and thus tempered but its derivative $e^x \cos(e^x)$ is not tempered.

Definition 6. *One can then define '$\mathcal{C}^\infty$-tempered' functions g as those for which g and all its derivatives are also tempered (space $\mathcal{O}_M(\mathbb{R}^n)$ chapter VII section 5 of [8]).*

Remark 4. All derivatives of the tempered distribution T of theorem 1 are themselves tempered if g is $\mathcal{C}^\infty$-tempered.

Physical fields have been early recognized as tempered distributions [7, 20, 21]. To a generalized field-function ϕ is then associated a continuous linear functional $\mathcal{S}(\mathbb{R}^n) \ni \rho \mapsto \Phi[\rho]$

$$\Phi[\rho] \equiv \langle \phi, \rho \rangle = \int_{\mathbb{R}^n} \mathrm{d}^n y \phi(\mathbf{y}) \rho(\mathbf{y}), \tag{2.1}$$

where we make the simplifying assumption of a smooth flat manifold covered by a single coordinate system (chart) and avoid the usual abuse of notation which identifies the distribution $\Phi[\rho]$ with the associated and eventually integrable field-function $\phi : \mathbb{R}^n \to \mathbb{C}$.

Definition 7. *For $x \in \mathbb{R}^n$ the translation operation τ_x defines the transformed distribution $\tau_x \Phi[\rho]$ according to*

$$\tau_x \Phi[\rho] = \langle \tau_x \phi, \rho \rangle \stackrel{\text{def}}{=} \langle \phi, \tau_{-x} \rho \rangle = \int_{\mathbb{R}^n} \mathrm{d}^n y \phi(y) \rho(y-x). \tag{2.2}$$

This is the value of the convolution $\Phi * \rho$ at the point x.

Remark 5. Let $(\rho_j)_{j\in\mathbb{N}}$ be a sequence of Dirac functions $\in \mathcal{S}(\mathbb{R}^n)$ (e.g. $x \mapsto \rho_j(x) = \frac{j^n}{\pi^{n/2}}\mathrm{e}^{-j^2r^2}$). When $j \to \infty$ the sequence converges to the reflection symmetric $\delta^n(x)$. Then if the mapping $\mathbb{R}^n \ni x \mapsto \phi(x)$ corresponds to a regular integrable field-function we have $\lim\limits_{j\to\infty} \Phi * \rho_j = \phi$. To potentially preserve this possibility we shall assume henceforth ρ to be a reflection symmetric test function of its argument.

The isomorphisms mentionned in relation to Fourier transforms permits to rewrite the convolution product in equation (2.2) as an integral in Fourier-space variables (Parseval–Plancherel relation)

$$(\Phi * \rho)(x) = \langle \tilde{\Phi}, \mathrm{e}^{(-i\ll .,x\gg)} f\rangle = \int_{\mathbb{R}^n} \frac{\mathrm{d}^n p}{(2\pi)^n}\, \mathrm{e}^{-\mathrm{i}\ll p,x\gg}\tilde{\phi}(p) f(p^2). \qquad (2.3)$$

Proposition 1. *For all distribution $\Phi \in \mathcal{S}'(\mathbb{R}^n)$ and all test function $\rho \in \mathcal{S}(\mathbb{R}^n)$*

(a) the convolution product $\Phi * \rho$ is in the class $\mathcal{C}^\infty$ on $\mathbb{R}^n$,
(b) For C^∞-tempered function g of theorem 1 the application $\varphi : \mathbb{R}^n \ni x \mapsto \varphi(x) = (\Phi*\rho)(x)$ defines a function in the class $\mathcal{S}(\mathbb{R}^n : \mathbb{C})$.
The proof is given in appendix A.

Examples.

(1) the proposition is trivial if the tempered distribution Φ is the Dirac mass δ, for $\mathcal{F}[\delta] = 1$ and $\mathcal{F}^{-1}[f] = \rho \in \mathcal{S}(\mathbb{R}^n)$;
(2) Regularisation of the characteristic distribution of a subset $A \subset \mathbb{R}^n$, e.g. associated to the function

$$\chi_A : \mathbb{R}^n \to \{0, 1\} \quad x \mapsto \chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

For $n = 1$ and $A = [0, 1]$ one has $\chi_A[\rho] = ((\mathbb{1} - \tau_{-1})H)[\rho] = H[(\mathbb{1} - \tau_1)\rho]$, where $H[\rho]$ is Heaviside's distribution associated to the function $H : \mathbb{R} \to \mathbb{R}$ defined by

$$H : \mathbb{R} \ni x \mapsto H(x) = \begin{cases} 0 & x \leqslant 0 \\ 1 & x > 0 \end{cases}$$

Then

$$\tau_x\chi_A[\rho] = \int_{\mathbb{R}} (H(y) - H(y-1))\rho(y-x)\,\mathrm{d}y \equiv \int_{x-1}^{x} \rho(y)\,\mathrm{d}y,$$

for ρ is reflection symmetric in y. Evidently

$$\partial_x[\tau_x\chi_A[\rho]] = \int_{\mathbb{R}} \partial_y[H(y) - H(y-1))]\rho(y-x),$$

and we are back to case 1 above, since $\partial_y H(y) = \delta(y)$.
(3) Regularisation of the Pseudo-function distribution (chapter II section 2 of [8]): For all $a \in \mathbb{R}$ we consider the function

$$x_+^a : \mathbb{R} \ni x \mapsto x_+^a(x) = \begin{cases} x^a & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$

This function is continuous on $\mathbb{R}^\star$, the space of linear forms on $\mathbb{R}$, where it is homogeneous of degree a and localy integrable for $a > -1$. It defines a distribution $\in \mathcal{S}'(\mathbb{R})$. For non-integer values of $a < -1$ the pseudo-function distribution $\in \mathcal{S}'(\mathbb{R})$, dubbed $Pf\, x_+^a$, is defined as

$$Pf\, x_+^a[\rho] = (-1)^k \int_0^\infty \frac{x^{a+k}}{(a+1)\cdots(a+k)} \rho^{(k)}(x)\,\mathrm{d}x,$$

for all test function $\in \mathcal{S}(\mathbb{R})$ and for all integer $k > -a - 1$. When treated in the sense of Hadamard's finite part this expression is independant of k [19]. For values $a \in \mathbb{C}$, $\Re(a) > -1$ this definition is extended to the distribution $\chi_+^a[\rho] = \frac{Pf\, x_+^a}{\Gamma(a+1)}[\rho] \in \mathcal{S}'(\mathbb{R})$, where Γ is Euler's function. It is easy to check that for $a \in \mathbb{R}\setminus \mathbb{Z}_-^\star$ $\left(\chi_+^a\right)' = \chi_+^{a-1}$. For negative integer values of a one has

$$\chi_+^{-k}[\rho] = \delta_0^{(k-1)}[\rho] = (-1)^{(k-1)}\rho^{(k-1)}(\mathbf{0}) \quad \text{if} \quad k > 1 \quad \text{(example 1)},$$
$$\chi_+^0[\rho] \;= H[\rho] \quad \text{(Heaviside's function on } \mathbb{R}_+^\star \text{, example 2)}.$$

Remark 6. In the situation of proposition 1(b) the function φ is regular integrable. Hence by remark 5 the associated distributions $\phi\left[\rho_j\right] \in \mathcal{S}'(\mathbb{R}^n : \mathbb{C})$ built on any sequence of Dirac test functions $\rho_j \in \mathcal{S}(\mathbb{R}^n)$ converge, when $j \to \infty$, to $\varphi(x)$. It is easy to see that the r-time iterated convolution $(((\Phi * \rho)) * \rho) * \cdots) * \rho$ corresponds also to a regular integrable field function taking the same form as φ in equation (2.3) with $f(p^2)$ replaced by $f^r(p^2)$. Thus we expect $\varphi(x)$ to remain unaffected after any arbitrary number of iterated convolution, to the extent that $f(p^2)$ and $f^r(p^2)$, for all positive integer r, all belong to $\mathcal{S}(\mathbb{R}^n)$. For instance the Dirac sequence $\lim_{j\to\infty} \mathcal{F}[\rho_j] = \lim_{j\to\infty} f_j = \mathbb{1}$, e.g. $f_j^r : p \mapsto f_j^r(p^2) = \mathrm{e}^{-r(\frac{p^2}{4j^2\Lambda^2})} \equiv \mathrm{e}^{-(\frac{p^2}{4j^2\Lambda^2})}$, for Λ is positive and arbitrary, that is $\frac{\Lambda^2}{r} \simeq \Lambda^2$.

As we shall see some of the regularity properties of $\varphi(x)$ can be studied after its localisation in an open subset Ω of $\mathbb{R}^n$, since test functions f exist that are unity on a compact of Ω (lemme 1.4.1 of [19]) with the seemingly evident local property $f^r = f$.

Some theorems valid for metrisable manifolds are particularly useful to clarify the situation.

Theorem 2. *Minkowskian and Euclidean manifolds being metrisable are paracompact with locally finite open coverings. Then there exists a partition of unity subordinate to any such open covering of the manifold.*

The proof can be found in various texts: for instance the first part of the assertion is in [22, 23], while the second part is in appendix D of [9].

Partitions of unity lead to localized distributions which can be *stock* together in a unique way according to

Theorem 3. *Let $(\Omega_j)_{j\in\mathbf{J}}$ be a family of open subset of $\mathbb{R}^n$ and $(T_j)_{j\in\mathbf{J}}$ be a family of distributions such that $T_j \in \mathcal{D}'(\Omega_j)$. The family $(T_j)_{j\in\mathbf{J}}$ is supposed to fulfil the compatibility condition $T_{j|\Omega_j\cap\Omega_k} = T_{k|\Omega_j\cap\Omega_k}$ for all $j, k \in \mathbf{J}$. Then there exists a single distribution T on $\Omega = \bigcup_{j\in\mathbf{J}} \Omega_j$ such that the restriction of T to each Ω_j is T_j.*

A proof is given in appendix B.

Remark 7. Each individual function, say $0 \leqslant f_j \leqslant 1$, composing the overall PU f is in the $\mathcal{C}^\infty$ class with $\mathrm{supp}(f_j) \subseteq$ closure of Ω_j. Hence $f \in \mathcal{D}(\Omega) = \bigcup_j \mathcal{D}(\Omega_j) \subset \mathcal{S}(\Omega)$. For abreviation purpose we dubbed f as PU Super Regular Test Functions (SRTF) to distinguish from alternative PU's which would involve annulation of only a limited number of derivatives on the closure of Ω . Evidently this category of test functions belongs to specific Sobolev Hilbert spaces with respect to $\mathcal{S}(\Omega)$. *A priori* the single distribution T of theorem 3 is of compact support and a fortiori a tempered distribution.

The properties just mentionned are commonly used in the definition of integrals of differential forms [24]. In a way similar to this case the classical field function $\varphi(x)$ of equation (2.3) is independent of the set up of this PU, since the f^r's of remark 6 stand now for the product of r different PU's which is just an other PU (cf appendix B for details). All definitions of the classical field function φ with different partition of unity are therefore equivalent, thereby eliminating the initial arbitrariness in the choice of test functions.

2.3. Classical Minkowskian fields as distributions

We turn now to specificities in relation with Minkowski physical space. Without loss of generality we may consider the free scalar field ϕ of positive mass m in a 4-dimensional Minkowski flat space $\mathcal{M}$ with coordinates $x = (x^0, x^1, x^2, x^3) = (x^0, \mathbf{x})$ (with a pseudo-metric $g_{\mu\nu} \stackrel{\text{def}}{=} \text{diag}\{1, -1, -1, -1\}$, with $\{\mu, \nu\} \in \{0, 1, 2, 3\}$; $\Box = \partial^\mu g_{\mu\nu} \partial^\nu$ and $\ll x, y \gg = x^\mu g_{\mu\nu} y^\nu$ the associated bilinear form). To the classical field-function $\phi(x^0, \mathbf{x})$ is then associated a distribution $\Phi[\rho]$ built on test functions $\rho : (\mathbb{R} \times \mathbb{R}^3) \ni (x^0, \mathbf{x}) \mapsto \rho(x^0, \mathbf{x}) \in \mathcal{S}(\bar{\mathbb{R}}^4)$ (the completed topological tensor product $\mathcal{S}(\mathbb{R})\hat{\otimes}\mathcal{S}(\mathbb{R}^3)$ [25]). It is important to notice that, due to separate reflexion symmetry of the test function ρ in the variables x^0 and $\mathbf{x}$, the translation-convolution product can still be written as an integral in Fourier-space variables with the proper pseudo-metric bilinear form, that is

$$(\Phi * \rho)(x^0, \mathbf{x}) = \langle \tilde{\Phi}, \mathrm{e}^{(-i\ll .,x\gg)} f \rangle = \int \frac{\mathrm{d}p_0\, \mathrm{d}^3\mathbf{p}}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}\ll p,x\gg} \tilde{\phi}(p^0, \mathbf{p}) f\big(p_0^2, \mathbf{p}^2\big).$$

The restriction of the classical field-function $\phi(x^0, \mathbf{x})$ to the hyperplane $(x = (x^0, \mathbf{x}) \in \mathcal{M} : x^0 = 0)$ is then described as an associated distribution on $\mathcal{S}'(\mathbb{R}^3)$ [10, 26]. The procedure involves the expansion of ϕ by changing the Klein–Gordon equation (KG-equation) $(\Box + m^2)\phi = 0$, via a Fourier transformation, into an algebric equation in the sense of distributions $(p^2 - m^2)\tilde{\phi}(\mathbf{p}) = 0$. This equation is solved by $\chi(\mathbf{p})\delta(p^2 - m^2)$, where $\chi(\mathbf{p})$ is some distribution associated to a function on $\Gamma_m = \{p \in \mathcal{M} : p^2 = m^2\}$. A precise definition of $\delta(p^2 - m^2)$ as a tempered distribution is given as a pullback of $\delta \in \mathcal{S}'(\mathbb{R})$ under the mapping $Q(p) = \mathbf{p}^2 - m^2 : \bar{\mathbb{R}}^4 \mapsto \mathbb{R}$ [10, 27]. It follows that

$$\varphi(x^0, \mathbf{x}) \equiv (\Phi * \rho)(x^0, \mathbf{x}) = \int \frac{\mathrm{d}p_0\, \mathrm{d}^3\mathbf{p}}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}\ll p,x\gg} \delta(p^2 - m^2)\chi(\mathbf{p}) f\big(p_0^2, \mathbf{p}^2\big). \tag{2.4}$$

Proposition 2. *$\varphi(x^0, \mathbf{x})$ obeys the KG-equation $(\Box + m^2)\varphi(x^0, \mathbf{x}) = 0$.*

The proof is direct: the test function $f\big(p_0^2, \mathbf{p}^2\big)$ ensures the existence of the integral and the KG-operator $(\Box_x + m^2)$ can be applied to $\mathrm{e}^{-\mathrm{i}\ll p,x\gg}$ to give a factor $(p^2 - m^2)\delta(p^2 - m^2) = 0$. Some more steps are necessary for the interpretation of $\varphi(x^0, \mathbf{x})$ as a candidate for the physical field.

Before carrying out the integration over p_0 one has to specify further χ in such a way that $\chi\delta(p^2 - m^2)$ is a well defined tempered distribution $\in \mathcal{S}'(\bar{\mathbb{R}}^4)$. It is well known ([27] and p 60 of [28]) that the general solution $\chi \in \mathcal{S}'(\bar{\mathbb{R}}^4)$ of $(p^2 - m^2)\chi = 0$ is of the form

$$\chi = \chi_+(\mathbf{p})\delta_+(p^2 - m^2) + \chi_-(\mathbf{p})\delta_-(p^2 - m^2)$$

with tempered distributions $\chi_\pm(\mathbf{p}) \in \mathcal{S}'(\mathbb{R}^3)$, where $\chi_\pm(\mathbf{p})\delta_\pm(p^2 - m^2) \in \mathcal{S}'(\bar{\mathbb{R}}^4)$ is defined by

$$\langle \chi_\pm \delta_\pm, f \rangle = \int \frac{\mathrm{d}^3\mathbf{p}}{2\omega(\mathbf{p})} \chi_\pm(\Omega_\pm(\mathbf{p})) f(\omega^2(\mathbf{p}), \mathbf{p}^2),$$

with $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$ and $\Omega_\pm(\mathbf{p}) = (\pm\omega(\mathbf{p}), \mathbf{p})$. In effect a localization of the distribution χ has occured, since $\mathcal{S}'(\bar{\mathbb{R}}^4)$ has been split into the duals of $\mathcal{S}(\bar{\mathbb{R}}_\pm)\hat{\otimes}\mathcal{S}(\mathbb{R}^3)$ as detailed in [25].

Then the classical field function $\varphi : (\mathbb{R} \times \mathbb{R}^3) \ni (x^0, \mathbf{x}) \mapsto \varphi(x^0, \mathbf{x}) \in \mathcal{S}(\bar{\mathbb{R}}^4)$ associated to the convolution product takes the final form

$$\varphi(x^0, \mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{f\big(\omega_p^2, \mathbf{p}^2\big)}{2\omega(\mathbf{p})} [\chi_+(\Omega_+(\mathbf{p}))\, \mathrm{e}^{-\mathrm{i}\ll\Omega_+(\mathbf{p}),x\gg} + \chi_-(\Omega_-(\mathbf{p}))\, \mathrm{e}^{-\mathrm{i}\ll\Omega_-(\mathbf{p}),x\gg}]. \quad (2.5)$$

2.4. OPVD formulation of the quantum field

Following the usual approach of canonical quantization the quantum scalar field operator $\hat{\varphi}(x^0, \mathbf{x})$ proceeds from equation (2.5) via the correspondance in terms of creation and annihilation operators $\{\chi_-(\Omega_-(\mathbf{p})) \curvearrowright a^+(\mathbf{p}), \chi_+(\Omega_+(\mathbf{p})) \curvearrowright a(\mathbf{p})\}$, with commutator algebra $[a(\mathbf{p}), a^+(\mathbf{q})] = (2\pi)^3 2\omega(\mathbf{p})\delta^3(\mathbf{p} - \mathbf{q})$. That is

$$\hat{\varphi}(x^0, \mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{f\big(\omega_p^2, \mathbf{p}^2\big)}{2\omega(\mathbf{p})} [a(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}\ll\Omega_+(\mathbf{p}),x\gg} + a^+(\mathbf{p})\, \mathrm{e}^{-\mathrm{i}\ll\Omega_-(\mathbf{p}),x\gg}]. \quad (2.6)$$

However, the one particle space $\mathfrak{H}_m$ of a relativistic spinless neutral particle (resp. antiparticle) of mass $m > 0$ in Minkowski space needs specifications. This is done in [27] and we only quote the main outcomes. Let $\Gamma_m^\pm = \{p \in \mathcal{M} : p^2 = m^2, \pm p^0 > 0\}$ be the smooth submanifolds of $\mathcal{M}$ associated to the decomposition present in equation (2.6). Then $\delta(p^2 - m^2)$ in equation (2.4) is identified with the positive Borel measure $d\mu_m$ on Γ_m and $\mathfrak{H}_m$ is $L^2(\Gamma_m^+, d\mu_m)$ (resp. $L^2(\Gamma_m^-, d\mu_m)$).

Remark 8. According to theorems 2 and 3 $f\big(\omega_p^2, \mathbf{p}^2\big) \in \mathcal{S}(\mathbb{R}^3)$ can be taken as a PU subordinate to the open covering of $\mathbb{R}^3 \supset \Omega = \cup_{j\in\mathbf{J}}(\Omega_j)$. The restriction of the distributions (cf appendix B) $\chi_\pm(\Omega_\pm(\mathbf{p}))$ to each Ω_j is defined in terms of the individual functions $0 \leqslant f_j \leqslant 1$, composing the overall PU f, as $\chi_\pm(\Omega_\pm(\mathbf{p}))\,|_{\Omega_j} = f_j\big(\omega_p^2, \mathbf{p}^2\big)\chi_\pm(\Omega_\pm(\mathbf{p}))$. Restrictions of creation and annihilation operators to each Ω_j are then defined accordingly.

If $|0\rangle$ is the vacuum state such that $a(\mathbf{p})|_{\Omega_j}|0\rangle = 0$, $\langle 0|0\rangle = 1$, theorem 3 permits to define the one particle state

$$a_f^+|_\Omega|0\rangle = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{f\big(\omega_p^2, \mathbf{p}^2\big)}{2\omega(\mathbf{p})} a^+(\mathbf{p})|_\Omega|0\rangle$$

and its norm $\|a_f^+|_\Omega|0\rangle\|^2_{L^2(\Omega)} = \langle 0|a_f|_\Omega a_f^+|_\Omega|0\rangle$

$$\begin{aligned}\|a_f^+|_\Omega|0\rangle\|^2_{L^2(\Omega)} &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{\overline{f\big(\omega_p^2, \mathbf{p}^2\big)}}{2\omega(\mathbf{p})} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{f\big(\omega_q^2, \mathbf{q}^2\big)}{2\omega(\mathbf{q})} \langle 0|[a(\mathbf{p})|_\Omega, a^+(\mathbf{q})|_\Omega]|0\rangle,\\ &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{\big|f\big(\omega_p^2, \mathbf{p}^2\big)\big|^2}{2\omega(\mathbf{p})}.\end{aligned}$$

This integral is bounded with a PU (cf appendix E).

To construct the full Hilbert space of states (Fock space) generated by vectors such that $|r\rangle = a_{f_1}^+|_\Omega a_{f_2}^+|_\Omega \cdots a_{f_r}^+|_\Omega|0\rangle$ we use the standard mathematical procedure of linear superposition and Cauchy's completion.

Its is crucial to verify that the definition (2.6) of the field-operator maintains relativistic invariance. It comes indeed as immediate corollaries:

(1) Lorentz invariance of the definition, for under a Lorentz transformation a scalar PU f changes to an equivalent scalar PU f',
(2) fulfilment of the Poincaré commutator algebra, for any product of PU's is an equivalent PU.

3. Extension of singular distributions, Lorentz invariance and BPHZ

3.1. Generalities on the extension of singular distributions.

Our concern is to give a mathematically well-defined meaning to a functional $A = \langle T(X), f(X)\rangle$ where $T(X)$ is a singular distribution and $f(X)$ a test function. X stands generically for a set of coordinates in ordinary or momentum space. $T(X)$ can have singularities either at some finite value X_0 or at infinity. Since the X_0-singularity can, by a change of variables, always be transported to $X = 0$, we treat in the following only the infrared case (singularity at $X = 0$) and the ultraviolet case (singularity at $X \to \infty$).

Because the functional integration of (2.1) may occur in practice in a space of smaller dimension than the original dimension D, we shall use d as a dimensional label henceforth. Let then $f(X)$ be a $\mathcal{C}^\infty$ test function belonging to $\mathcal{S}(\mathbb{R}^d)$ and $T(X)$, a distribution belonging to $\mathcal{S}'(\mathbb{R}^d\backslash\{0\})$ which we want to extend to the whole domain $\mathcal{S}'(\mathbb{R}^d)$. The singular order k of $T(X)$ at the origin of $\mathbb{R}^d$ is defined as

$$k = \inf\{s : \lim_{\lambda\to 0}\lambda^s T(\lambda X) = 0\} - d.$$

In order to achieve the extension, one usually performs an educated Taylor surgery on the test function by throwing away the weighted k-jet of $f(X)$ at $X = 0$. Denoting the resulting Taylor remainder by $R^k f(X)$, one obtains

$$R^k f(X) = f(X) - \sum_{n=0}^{k}\sum_{|\alpha|=n}\frac{(X)^\alpha}{\alpha!}\partial^\alpha f(X)|_{X=0}. \tag{3.1}$$

It is assumed that the test function also takes care of all other singularities eventually present in $T(X)$.[3] Now the extension $\tilde{T}(X)$ of $T(X)$ can be defined by transposition:

$$\langle\widetilde{T}, f\rangle := \langle T, R^k f\rangle. \tag{3.2}$$

The extension $\tilde{T}(X)$ so obtained is not unique. We shall come back to this point. As to the definition of $\tilde{T}(X)$, there is an abundant literature on the original procedure of [6] and on important improvements proposed recently [29, 30]. Our concern here [31] is the use of SRTFs to carry out the extension. With a SRTF, one has strictly $f(X) = R^k f(X)$, and therefore $R^k f(X)$ also belongs to $\mathcal{S}(R^d)$. Then the following set of equalities holds:

$$\langle T, f\rangle = \langle T, R^k f\rangle = \langle R^k T, f\rangle \equiv \langle\widetilde{T}, f\rangle. \tag{3.3}$$

As we shall see in the sequel, important consequences follow from these relations. They have to do with the splitting procedure of distributions into advanced and retarded parts, Lorentz covariance of the results, the connection with BPHZ renormalization and consequently the analysis of the IR and UV behaviour of the underlying QFT. We shall treat the IR and the UV case separately.

3.2. Lorentz invariance and super-regular test functions

In the Taylor surgery context of the Epstein–Glaser (EG) procedure under the action of an element Λ of the Lorentz group in the general Taylor expansion, one has $X^\alpha\partial_\alpha(\Lambda f)(0) = (\Lambda^{-1}X)^\beta\partial_\beta f(0)$. Lorentz invariance is therefore violated in this procedure. However, as mentioned earlier, $\widetilde{T}(X)$ is determined up to a sum of derivatives of δ functions $\sum_{|\alpha|\leqslant k}(-1)^\alpha\frac{a_\alpha}{\alpha!}\delta^{(\alpha)}(X)$. But these δ-terms transform as $\delta^{(\alpha)}(\Lambda X) = [\Lambda^{-1}X]^\alpha_\beta\delta^{(\beta)}(X)$. EG's remedy is then to determine the a_α's to correct for the violation due to derivatives of f. With

[3] This is not actually the case of $R^k f(X)$ in (3.1), for it does not belong to $\mathcal{S}(\mathbb{R}^d)$, the space of Schwartz's test functions [6].

SRTF vanishing at the origin with all their derivatives, because of the identity $f(X) \equiv R^k f(X)$, Lorentz violating derivatives are just not there. Provided then that the space of test functions to be used is restricted to SRTF types, Lorentz invariance is satisfied from the outset. Formally $\widetilde{T}(X)$ is only determined up to the sum of derivatives of δ-terms, but their contributions are redundant with super-regular test functions. Nevertheless, the presence of Lorentz-invariant terms in this sum should not be forgotten and may prove instrumental in restoring some other broken symmetries [32, 33].

3.3. Link with BPHZ renormalization

One of the obstacles for a widespread application of the EG approach was the non-evident link to other renormalization schemes and in particular the question of its multiplicative structure and its feasibility in renormalization group studies. This question was discussed in [29, 30] and more recently in [5]. As we shall see, the introduction of PU-SRTFs permits us to simplify and extend somewhat the analyses made in these studies. As a matter of fact, the chain of identities of (3.3) valid for SRTF allows us to establish quite simply the link of our approach with BPHZ renormalization. In the approach with SRTF, the Fourier transform of the singular $T(X)$ is well defined since the functional built with SRTF identical to their Taylor remainder itself is well defined. Therefore, the following chain of equalities holds:

$$\begin{aligned}\langle T, f\rangle &= \langle T, R^k f\rangle = \langle \mathcal{F}(T), \mathcal{F}^{-1}(R^k f)\rangle = \langle \mathcal{F}(T), R^k(\mathcal{F}^{-1}(f))\rangle \\ &= \langle R^k \mathcal{F}(T), \mathcal{F}^{-1}(f)\rangle,\end{aligned} \tag{3.4}$$

that is,

$$\mathcal{F}(\tilde{T}) = R^k \mathcal{F}(T). \tag{3.5}$$

This is BPHZ subtraction at zero momentum, up to, as is well known, an arbitrary polynomial in p originating from the sum of δ-terms mentioned previously. It is known that for a non-massive QFT, the BPHZ method faces infrared divergences. However, a mass scale can be introduced by doing subtractions at some external momenta $q \neq 0$. For non-SRTF, it amounts to choosing a weight function—in front of the sums in (3.1)—such that $w(X) = \mathrm{e}^{\mathrm{i}qX} \notin \mathcal{S}(R^D)$, the space of Schwartz test functions[4]. However, the situation is much simpler with SRTF. One has still $\mathrm{e}^{\mathrm{i}qX} f(X) \equiv R^k(\mathrm{e}^{\mathrm{i}qX} f(X))$, and the chain of equalities in (3.4) can be rewritten with this modification of f, leading now to BPHZ subtraction at an arbitrary momentum $p = q$. Thus, BPHZ appears just as a special case of the EG method, or in other words: the validity of BPHZ is a corollary of TLRS. From a look at Zimmermann's solution of the recursion equation in the BPHZ context, one notices that the counterpart of the notion of a superficially divergent subdiagram—used in the definition of forests—is the statement that a singular distribution T has to be replaced by an extension $\tilde{T}$. However, to calculate a multiloop diagram with overlapping parts, it is not necessary to go through the cumbersome machinery of the BPHZ method. For instance, a useful alternative technique of evaluation is the use of the parametric representation of Feynman amplitudes [34]. An example of this type is presented in appendix C. In the next section, we examine in more details UV and IR behaviours when using PU test functions introduced in section 2.

[4] Non-unique interpretations of diverging integrals must then be introduced [30].

4. UV and IR extensions

4.1. UV problematics and PU with running boundaries

To present the PU boundary problem in simple terms in the UV, we shall consider a physical amplitude $\mathcal{A}$ in one dimension written schematically as

$$\mathcal{A} = \int_0^\infty \mathrm{d}X T(X) f(X), \tag{4.1}$$

where $f(X)$ is a test function of finite extension—or finite support—vanishing with all its derivatives at its boundaries. In this form, the amplitude $\mathcal{A}$ does not differ from the calculation using a cut-off procedure. In the UV domain, for example, the cut-off, denoted by H, would correspond to the support of f, with $f(X \geqslant H) = 0$. Then if $\lim_{X\to\infty} T(X) \simeq \frac{1}{X}$, $\mathcal{A}$ would diverge with H as $\log(H)$. Using a PU this way would then be a useless artefact. However, a PU is not just a way of constructing an overall cut-off function of value 1 almost everywhere, for this can be done in many ways without ever referring to PUs. But a PU is *'subordinate to a specific open covering of the manifold'*, a property wiped out when using a sharp cut-off. We shall show that it is this property which permits us to go beyond the brutal sharp cut-off through the introduction of running boundaries. The procedure is detailed in [14, 31, 35], and here we give only the necessary explanations in relation with the crucial *'subordinate'* property.

In QFT, $T(X)$ is in general a distribution in the variable $\|X\|$ only[5]. We set $\|X\| \equiv X$ from now on. A PU on some finite support can be built up from a basic super regular function $u(X)$ such that

$$u(X) + u(h - X) = 1 \quad \text{for any} \quad X \in [0, h], \tag{4.2}$$

where h is a positive real number, and take

$$\beta_i(X) \equiv u(|X - \mathrm{i}h|) \quad \text{for} \quad |X - \mathrm{i}h| < h. \tag{4.3}$$

Then a PU is obtained as

$$f_{\mathrm{PU}}(X) = \sum_{i=1}^{N} \beta_i(X), \tag{4.4}$$

with the following properties:

$$f_{\mathrm{PU}}(X) = \begin{cases} u(h - X) & \text{for} \quad X \in [0, h] \\ 1 & \text{for} \quad X \in [h, Nh] \\ u(X - Nh) & \text{for} \quad X \in [Nh, (N+1)h]. \end{cases} \tag{4.5}$$

The covering of the space (here a line segment) is therefore accomplished by the sum on subspaces Ω_i to which the β_i are subordinate. It is important to note that the super regular properties as well as relation (4.2) for the functions β_i are preserved when h depends on X. For later purposes, it is convenient to parametrize $h(X)$ as follows:

$$h(X) = \eta^2 X g_\alpha(X) + (\alpha - 1), \tag{4.6}$$

where η and α are positive parameters. With this choice of $h(X)$, it is sufficient to consider only two functions β_i, with $i = 1, 2$, to build up the simplest PU. β_1 and β_2 vanish respectively with all their derivatives when $X = h(X)$ and $X = 2h(X)$. We demand that when $\alpha \to 1^-$,

[5] We shall consider the general situation in the last section.

the sum of the two β's cover the whole integration domain in X. In the small X region (IR), it is easy to see that the condition

$$\lim_{\alpha\to 1^-} g_\alpha(X) = 1 + \mathcal{O}((\alpha-1)X), \tag{4.7}$$

leads to the desired effect when solving the two equations $X = 2h(X)$ and $X = h(X)$ for small X. Indeed the solutions are respectively $X01 = \frac{2(1-\alpha)}{2\eta^2-1}$ and $X02 = \frac{(1-\alpha)}{\eta^2-1}$, which both tend to 0^+ when $\eta^2 > 1$ and $\alpha \to 1^-$. $X01$ being the lower boundary of the PU-support, the lower limit of the X-integration tends to zero as it should. Moreover, and whatever the functional form of $u(X)$, the resulting PU behaves like a θ-function in the IR. In the large X limit (UV), any functional form of $g_\alpha(X)$ satisfying condition (4.7) will do, and we keep the choice made earlier in [31, 35], that is $g_\alpha(X) = X^{(\alpha-1)}$. Other choices modify only the rate at which the upper boundary of the PU-support goes to infinity when α approaches 1^-, the extended distribution being the same, as it should. Solving again the two equations $X = h(X)$ and $X = 2h(X)$ in the large X limit—where $(\alpha-1)$ can be neglected in (4.6)—the solutions are respectively $X11 = (\eta^2)^{(\frac{1}{1-\alpha})}$ and $X12 = (2\eta^2)^{(\frac{1}{1-\alpha})}$. For $\eta^2 > 1$, both values tend to infinity when $\alpha \to 1^-$, and the PU-support stretches to cover the whole domain of integration in X. However, it does so in a remarkable way: the PU is 1 up to the point $X11$ and the region where it falls from 1 to 0 increases proportionally to $X11$ when $X11$ goes to infinity, implying an infinitesimal drop-off in the asymptotic limit. We call this behaviour an ultrasoft cut-off. All those features are generic of the PU set-up since they are independent of the functional form of $u(X)$ and of the value of the scale parameter $\eta > 1$ governing the shape of β_1 and β_2 and the size of their respective support[6].

Going back to the UV analysis of a general QFT amplitude $\mathcal{A}$, for its PU and with $f(X) \equiv f^>(X)$, it holds that

$$\begin{aligned} f^>(X) &\equiv -\frac{1}{k!}\int_1^\infty \mathrm{d}t(1-t)^k\partial_t^{(k+1)}[t^k f^>(Xt)], \\ &= -\frac{X}{k!}\int_1^\infty \frac{\mathrm{d}t}{t}(1-t)^k\partial_X^{(k+1)}[X^k f^>(Xt)]. \end{aligned} \tag{4.8}$$

For $f^>(tX)$ present in Lagrange's formula, one has $t < \frac{h(X)}{X} = \eta^2 g(X)$. Then

$$\begin{aligned} \langle T, f^>\rangle &= \int \mathrm{d}^d X T(X)\left\{-\frac{X}{k!}\int_1^{\eta^2 g(\|X\|)} \frac{\mathrm{d}t}{t}(1-t)^k\partial_X^{(k+1)}[X^k f^>(Xt)]\right\} \\ &= \langle \widetilde{T}^>, f^>\rangle, \end{aligned} \tag{4.9}$$

where in the last line partial integrations on X have been performed. For the UV-extension $\widetilde{T}^>(X)$ of $T(X)$, one finally obtains

$$\widetilde{T}^>(X) = \frac{(-)^k}{k!}X^{(k-d+1)}\partial_X^{(k+1)}[X^d T(X)]\int_1^{\eta^2}\frac{\mathrm{d}t}{t}(1-t)^k. \tag{4.10}$$

Here the limit $\alpha \to 1^-$ has been taken under the proviso that k is such that the d-dimensional integral on X of $\widetilde{T}^>(X)$ alone is finite. Independent of the choice of $g_\alpha(X)$, the scale parameter η will always be present in the extended distribution $\tilde{T}$. In this sense, η is a universal parameter with essential relevance for renormalization group studies. We shall see that η is closely related to the arbitrary scale introduced in dimensional regularization.

The form (4.10) is not unique. Other forms can be obtained via a change of variables in the integrals (4.9). Fore instance, the replacement $Xt \to Y$ leads to

$$\widetilde{T}^>(X) = \frac{(-1)^k}{k!}X^{(k-d+1)}\partial_X^{(k+1)}\left[X^d\int_1^{\eta^2}\mathrm{d}t\frac{(1-t)^k}{t^{(d+1)}}T\left(\frac{X}{t}\right)\right]. \tag{4.11}$$

[6] Figure 6 of [35] shows a PU set-up for a particular functional choice of the elementary function $u(X)$.

The extension of $\widetilde{T}^{>}(X)$ from $T(X)$ works then via subtractions as demonstrated in previous publications [14, 31, 35]. Of course, all these different forms do not change the value of the functional in (4.9), though the corresponding extensions can be quite different. For example, in dimension 1, the extension of $\frac{1}{X+a}$ is $\frac{1}{X+a} - \frac{1}{X+\eta^2}$ and the extension of the $2d$-Euclidian propagator $\frac{1}{p^2+m^2}$ is $\frac{1}{p^2+m^2} - \frac{1}{p^2+\eta^2 m^2}$. The number of subtractions depends on the degree of the UV-divergence and increases with their degree. The subtractions look formally like Pauli–Villars subtractions, but their physical content is completely different. The scale parameter η^2 is finite but arbitrary—which is essential for renormalization group studies. On the other hand, the Pauli–Villars masses have no physical *'raison d'être'* and have finally to be sent to infinity.

4.2. IR extensions

To avoid confusion, now we shall write $f(X) \equiv f^{<}(X)$. We showed previously that in the IR, $f^{<}(X)$ becomes a θ-function. In order to keep track of the explicit regularization by the test function of the $X = 0$ singularity, we may study the amplitude in terms of $[f^{<}]^2 \sim f^{<}$, where $[f^{<}]^2$ stands for the product $f_1^{<} f_2^{<}$ of two PUs as discussed in subsection (2.1). It is important to note that the two PUs do not need to have the same η dependence, indicating thereby that the IR and UV treatments are in fact completely decoupled. We shall use Lagrange's multi-dimensional formula for one of the two $f^{<}$'s in a form suited for the IR domain:

$$R^k f_2^{<}(X) = (k+1) \sum_{|\beta|=k+1} \left[\frac{X^\beta}{\beta!} \int_0^1 \mathrm{d}t (1-t)^k \partial_{(tX)}^\beta f_2^{<}(tX) \right]. \tag{4.12}$$

The notations are the same as in (3.1). One has then

$$\langle T, [f^{<}]^2 \rangle = (k+1) \sum_{|\beta|=k+1} \int \mathrm{d}^d X T(X) f_1^{<}(X) \left[\frac{X^\beta}{\beta!} \int_0^1 \mathrm{d}t (1-t)^k \partial_{(tX)}^\beta f_2^{<}(tX) \right], \tag{4.13a}$$

$$= \langle T f_1^{<}, R^k f_2^{<} \rangle. \tag{4.13b}$$

By a change of variable $Xt \to X$, this becomes, after partial integration on X,

$$\begin{aligned} \langle T, [f^{<}]^2 \rangle &= (-1)^{(k+1)} (k+1) \sum_{|\beta|=k+1} \int \mathrm{d}^d X \partial_X^\beta \left[\frac{X^\beta}{\beta!} \int_0^1 \mathrm{d}t T\left(\frac{X}{t}\right) f_1^{<}\left(\frac{X}{t}\right) \right] \frac{(1-t)^k}{t^{d+k+1}} f_2^{<}(X) \\ &= \langle \widetilde{T}^{<}, f_2^{<} \rangle. \end{aligned} \tag{4.14}$$

In the region where the test function is less than 1, we have effectively $(1-\alpha)\|X\| > h((1-\alpha)\|X\|)$. This defines the lower limit for the t integration in (4.14), with $(1-\alpha)\frac{\|X\|}{t} > h\big((1-\alpha)\frac{\|X\|}{t}\big)$, i.e. $t > \|X\|(\eta^2 - 1) = \tilde{\eta}\|X\|$. Taking the limit where the test function extends to one in the whole space, we have then[7]

$$\widetilde{T}^{<}(X) = (-1)^{(k+1)} (k+1) \sum_{|\beta|=k+1} \partial_X^\beta \left[\frac{X^\beta}{\beta!} \int_{\tilde{\eta}\|X\|}^1 \mathrm{d}t T\left(\frac{X}{t}\right) \right] \frac{(1-t)^k}{t^{d+k+1}}. \tag{4.15}$$

Note that the derivatives in (4.15) and (4.16) below have to be taken in the sense of distributions. The scale $\tilde{\eta}$ is positive and arbitrary.

[7] Up to a an arbitrary sum of terms proportional to the derivatives of δ-distributions at the singularity $\delta^{(\alpha)}(X)$ with $|\alpha| = k$.

For a homogeneous distribution, with $T[\frac{X}{t}] = t^{(k+d)}T(X)$, the t integration can be carried out to give (cf appendix D)

$$\widetilde{T}^{<}(X) = (-1)^k(k+1)\sum_{|\alpha|=k+1}\partial_X^\alpha\left[\frac{X^\alpha}{\alpha!}T(X)\log(\tilde{\eta}\|X\|)\right] + H_k\sum_{|\alpha|=k}\frac{(-1)^{|\alpha|}}{\alpha!}C^\alpha\delta^{(\alpha)}(X), \tag{4.16}$$

with

$$H_k = \sum_{p=1}^{k}\frac{(-1)^{p+1}}{p}\binom{k}{p} = \gamma + \psi(k+1) \qquad \text{and} \qquad C^\alpha = \int_{(X=1)} T(X)X^\alpha\,\mathrm{d}S$$

and ψ is the usual Digamma function with $\psi(1) = -\gamma$.

The extension $\widetilde{T}^{<}(X)$ differs from the original distribution $T(X)$ only at the singularity. This result was first obtained in [30], although under quite different premisses. It shows clearly how the use of PU-SRTF permits us to go beyond the general but incomplete IR analysis of subsection 3.3. An important check of the relevance of relation (4.16) is the infinite and exact resummation of the conventionally hopeless perturbative mass expansion of the scalar field theoretic propagator [30, 31]. Relation (4.16) is also essential to the treatment of the long-standing problem of singularities in the gauge-boson propagator in the light-cone gauge [36].

Up to now, it was assumed that the test functions were functions of x^2 or p^2 only. This is all right for applications in the Euclidean case. But things are technically more involved in Minkowski or light-cone metric.

In the first case, the test function becomes a function $\rho(x_0, \vec{x})$ in coordinate space or $f\left(p_0^2, \vec{p}^2\right)$ in momentum space. As usual, it is preferable to first carry out the integrations over x_0 or p_0, respectively, and then the remaining integrals. There are two possible situations.

(1) The integrals over x_0 or p_0 are convergent. In this case, the remaining integrations are as in the Euclidean space with dimension $D-1$ and the extension is calculated correspondingly.

(2) The integrals over x_0 or p_0 diverge. This requires an extension of the distribution with respect to the dependence on x_0 or p_0.

For practical reasons, the calculations are usually done in momentum space, and we restrict the discussion to this case. In principle, there can be an UV-divergence problem of the p_0-integration or there can be nonintegrable poles at finite values of p_0. The latter singularities are of IR-nature. Such situations can arise when extending non-causal distributions like powers of Feynman propagators. Once the p_0-extended distribution has been calculated, the remaining integrations can be done as in case (1). Similar considerations are valid for light-cone quantization with the replacements $x_0 \to x_+$ and $p_0 \to p_-$. An example for this type of calculations in the Minkowski case is given in appendix E. Examples for the light-cone case can be found in [14, 35].

5. Causal splitting of distributions and link with dispersion relation techniques

5.1. Causal splitting with test functions

For many applications, it is necessary to perform a time ordering in distributions depending on many different time arguments. Usually the time ordering is made introducing appropriate θ-functions of differences of time arguments. However, in general, this is not justified because of the appearance of ill-defined products of distributions leading to divergences. The way out of these problematics has been discussed extensively in the literature in the framework of the EG approach [32].

Here we want to discuss the splitting procedure in the TLRS framework. The ordering operation requires to split distributions T into advanced and retarded parts T_a and T_r, respectively, in the following way

$$T(x_1, \ldots, x_n) = T_r(x_1, \ldots, x_n) - T_a(x_1, \ldots, x_n). \tag{5.1}$$

The support of $T_r(x_1, \ldots, x_n)$, $(T_a(x_1, \ldots, x_n))$ is the n-dimensional generalization of the closed forward (backward) cone of $\{x_1, \ldots, x_n\}$. Because of translational invariance, one can put $x_n = 0$. If $T(x_1, \ldots, x_n)$ were regular at $x_i = 0$, $i = 1, \ldots, n-1$, the splitting could be performed with

$$T_r(x_1, \ldots, x_n) = \prod_{i=1}^{n-1} \theta\big(x_j^0 - x_n^0\big) T(x_1 - x_n, \ldots, x_{n-1} - x_n, 0),$$
$$T_a(x_1, \ldots, x_n) = \prod_{i=1}^{n-1} \theta\big(x_n^0 - x_j^0\big) T(x_1 - x_n, \ldots, x_{n-1} - x_n, 0).$$

On the contrary, if the product of θ-functions with the distribution T is ill-defined, the splitting procedure has to be done with the help of (3.3). The defining equations of $\widetilde{T}_r$ and $\widetilde{T}_a$ are then

$$\begin{aligned} \langle T, \theta f\rangle &= \langle \theta \widetilde{T}, f\rangle = \langle \widetilde{T}_r, f\rangle; \\ \langle T, (1-\theta) f\rangle &= \langle (1-\theta)\widetilde{T}, f\rangle = \langle \widetilde{T}_a, f\rangle. \end{aligned} \tag{5.2}$$

$\widetilde{T}_r$ and $\widetilde{T}_a$ are simply obtained by multiplication of $\widetilde{T}$ with the corresponding θ-functions. With the original singular distribution T, this would not have been possible. Finally, the following identification is obtained:

$$\widetilde{T}_r = \theta \widetilde{T}; \qquad \widetilde{T}_a = (1-\theta)\widetilde{T}. \tag{5.3}$$

It should be kept in mind that $\widetilde{T}_r$ and $\widetilde{T}_a$ are not unique, since, as stated above, $\widetilde{T}$ is not unique.

If $T(X) = T(x_1, \ldots, x_n)$ is causal, then the products of θ-functions in (5.2) allow an essential simplification particularly useful for calculations in momentum space [32]. A $4(n-1)$-dimensional vector $v = (v_1, v_2, \ldots, v_{n-1})$ is defined, made of $n-1$ time-like four-vectors all lying in the interior of the forward light-cone. The scalar product $v_i \cdot X_i = v_i^0 X_i^0 - \overrightarrow{v} \cdot \overrightarrow{X}$ is used to define the scalar $v \cdot X = \sum_{j=1}^{n-1} v_j \cdot X_j$, where one puts $X_n = 0$. Then $v \cdot X$ has the following properties:

$v \cdot X > 0$, $\forall X_j$ in the forward light-cone Γ^+;

$v \cdot X < 0$, $\forall X_j$ in the backward light-cone Γ^-.

Consequently, $v \cdot X = 0$ defines an hyperplane that separates the causal support into advanced and retarded parts.

The following indentities then hold:

$$\begin{aligned} \theta(v \cdot X) &= \prod_{j=1}^{n-1} \theta\big(X_j^0 - X_n^0\big); \qquad X \in \Gamma^+ \\ \theta(v \cdot X) &= \prod_{j=1}^{n-1} \theta\big(X_n^0 - X_j^0\big); \qquad X \in \Gamma^-. \end{aligned} \tag{5.4}$$

These identities can only be used with distributions having causal supports. Otherwise, the scalar products $v_i \cdot X_i$ would not have a unique sign related to advanced or retarded coordinates and $v \cdot X = 0$ could not be used to distinguish advanced and retarded parts of the supports. For later use, we need the Fourier transform of $\theta(v \cdot X)$, i.e. $\overline{\theta_v}(p)$, where $p = (p_1, p_2, \ldots, p_{n-1})$. This multi-dimensional quantity can be reduced to a four-dimensional one by

(1) choosing a coordinate system for which $p = (p_1, 0, \ldots, 0)$, where $p_1 = \big(p_1^0, \overrightarrow{p_1}\big)$ (which can be obtained by an orthogonal transformation in $\mathbb{R}^{4(n-1)}$);

(2) and choosing $v = (v_1, 0, \ldots, 0)$; $v_1 = (1, \overrightarrow{v})$ which yields $\theta(v \cdot X) = \theta\big(X_1^0 - \overrightarrow{X_1} \cdot \overrightarrow{v}\big)$.

Then

$$\overline{\theta_v}(p) = \frac{1}{(2\pi)^{D/2}} \int \mathrm{d}^D X_1 \, \mathrm{e}^{\mathrm{i}p \cdot X_1} \theta\big(X_1^0 - \overrightarrow{X_1} \cdot \overrightarrow{v}\big). \tag{5.5}$$

Now with

$$\theta\big(X_1^0 - \overrightarrow{X_1} \cdot \overrightarrow{v}\big) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\tau \frac{\mathrm{e}^{\mathrm{i}\tau(X_1^0 - \overrightarrow{X_1} \cdot \overrightarrow{v})}}{\tau - \mathrm{i}\epsilon},$$

one obtains

$$\overline{\theta_v}(p) = (2\pi)^{(D/2-1)} \frac{1}{p^0 + \mathrm{i}\epsilon} \delta^{(D-1)}(\overrightarrow{p} - p^0 \overrightarrow{v}). \tag{5.6}$$

The last equality means that $\overrightarrow{p}$ and $\overrightarrow{v}$ come out parallel to each other. Of course, physical results should not depend upon the choice of the arbitrary vector v. This will be verified in the next subsection.

5.2. Link with dispersion relations

As an application of the previous section, we work out explicitly the calculations of the retarded/advanced extension of a singular distribution. The result shows an interesting link to subtraction techniques known from dispersion relations.

The starting point is the following form of Lagrange's formula for SRTFs:

$$f^<(X) = (\omega + 1) \sum_{|\beta| = \omega+1} \left[\frac{X^\beta}{\beta!} \int_0^1 \mathrm{d}t \frac{(1-t)^\omega}{t^{(\omega+1)}} \partial^\beta_{(X)} f^<(Xt) \right]. \tag{5.7}$$

It yields $\widetilde{T_r^<}(X)$ after partial integrations and taking into account the restriction brought about by $f^<(Xt)$ in the t-integral:

$$\widetilde{T_r^<}(X) = (-1)^{(\omega+1)} \sum_{|\beta| = \omega+1} \partial_X^\beta \left[\frac{X^\beta}{\beta!} \int_0^1 \mathrm{d}t \frac{(1-t)^\omega}{t^{(d+\omega+1)}} \theta\left(\frac{v.X}{t}\right) f^<\left(\frac{X}{t}\right) T\left(\frac{X}{t}\right) \right]. \tag{5.8}$$

However, in keeping with the PV-type of subtraction, we shall see below that it also provides the interpretation of the subtraction in the dispersion relations. The corresponding Fourier transform—which becomes UV-regulated because in the process the couple $\{X01, X02\}$ gives by inversion the couple $\{p11, p12\}$ in p-space which provides a similar UV regulation as that discussed originally, hence its notation—is

$$\overline{\widetilde{T_r^>}}(p) = \frac{(\omega+1)}{(2\pi)^{d/2}} \sum_{|\beta| = \omega+1} \left[\frac{p^\beta}{\beta!} \int_{1/\eta^2}^1 \mathrm{d}t \frac{(1-t)^\omega}{t^{(\omega+1)}} \int \mathrm{d}k \overline{\theta_v}(k) \partial_p^\beta \overline{T}(pt - k) \right]. \tag{5.9}$$

Using (5.6) for $\overline{\theta_v}(k)$, this becomes

$$\overline{\widetilde{T_r^>}}\big(p_1^0\big) = \frac{\mathrm{i}}{2\pi} \frac{\big(p_1^0\big)^{\omega+1}}{\omega!} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_1^0}{k_1^0 + \mathrm{i}\epsilon} \int_{1/\eta^2}^1 \mathrm{d}t (1-t)^\omega \partial^{(\omega+1)}_{(p_1^0 t)} \overline{T}\big(p_1^0 t - k_1^0, 0, \ldots, 0\big). \tag{5.10}$$

With the change of variables $p_1^0 t - k_1^0 = k_1'^0$ and after $\omega + 1$ partial integrations on $k_1'^0$, one obtains

$$\overline{\widetilde{T_r^>}}\big(p_1^0\big) = \frac{\mathrm{i}}{2\pi} \frac{\big(p_1^0\big)^{\omega+1}}{\omega!} (1-\eta^2)^{(\omega+1)} \int_{-\infty}^{\infty} dk_1'^0 \frac{\overline{T}\big(k_1'^0, 0, \ldots, 0\big)}{\big(p_1^0 - k_1'^0 \eta^2 + \mathrm{i}\epsilon\eta^2\big)^{(\omega+1)} \big(p_1^0 - k_1'^0 + \mathrm{i}\epsilon\big)}, \tag{5.11}$$

where the final t-integration has been performed to give

$$\int_{1/\eta^2}^{1} \mathrm{d}t \frac{(1-t)^{\omega}}{\left(p_1^0 t - k_1'^0 + \mathrm{i}\epsilon\right)^{(\omega+2)}} = \frac{(\eta^2-1)^{(\omega+1)}}{(\omega+1)\left(p_1^0 - k_1'^0 + \mathrm{i}\epsilon\right)} \frac{1}{\left(p_1^0 - k_1'^0\eta + \mathrm{i}\epsilon\eta^2\right)^{(\omega+1)}}. \tag{5.12}$$

Finally with the substitution $k_1'^0 = sp_1^0$ in equation (5.11) and returning to the general frame [32], we obtain

$$\widetilde{\overline{T_r^{>}}}(p) = \frac{\mathrm{i}}{2\pi} \frac{(\eta^2-1)^{(\omega+1)}}{(\eta^2)^{(\omega+1)}} \int_{-\infty}^{\infty} \mathrm{d}s \frac{\overline{T}(sp)}{(s-1/\eta^2-\mathrm{i}\epsilon')^{(\omega+1)}(1-s+\mathrm{i}\epsilon')}. \tag{5.13}$$

The retarded, extended distribution $\widetilde{\overline{T_r^{>}}}(p)$ satisfies an unsubtracted dispersion relation since $\widetilde{T_r^{<}}(X)$ of equation (5.1) is a well-defined regular quantity. On the other hand, the original retarded distribution $T_r^{<} = \theta T$ is singular. A dispersion relation for this quantity would require $\omega+1$ subtractions. It is interesting to see that the explicit definition of the extended distribution in terms of the original singular distribution leads to the factor $(s-1/\eta^2-\mathrm{i}\epsilon')^{(\omega+1)}$ in the denominator of equation (5.13) which is characteristic of dispersion relations with $\omega+1$ subtractions, with one important difference: the subtraction point is not arbitrary but $\frac{p}{\eta^2}$, as shown hereafter. It is the scale η present in the SRTF which fixes this point. The calculation of the advanced, extended distribution $\widetilde{\overline{T_a^{>}}}(p)$ follows the same lines with a result similar to that of equation (5.13) with $\mathrm{i}\epsilon \to -\mathrm{i}\epsilon$. The difference of the retarded and advanced pieces reduces to

$$\begin{aligned}\widetilde{\overline{T_r^{>}}}(p) - \widetilde{\overline{T_a^{>}}}(p) &= \overline{T}(p) - \sum_{n=0}^{\omega} \frac{(-1)^n}{n!}\left[\frac{\eta^2-1}{\eta^2}\right]^n \int_{-\infty}^{\infty} \mathrm{d}s\overline{T}(sp)\delta^{(n)}\left(s-\frac{1}{\eta^2}\right),\\ &= \overline{T}(p) - \sum_{n=0}^{\omega}\sum_{|\beta|=n} \frac{\left[\frac{p(\eta^2-1)}{\eta^2}\right]^{\beta}}{\beta!}\overline{T}^{(\beta)}\left(\frac{p}{\eta^2}\right),\end{aligned} \tag{5.14}$$

which is just the Taylor remainder of $\overline{\overline{T}}(p-q)$ at $q = \frac{p}{\eta^2}$, that is, BPHZ subtraction at $q = \frac{p}{\eta^2}$ discussed in subsection (3.3).

6. Conclusions

The axiomatic approach towards a rigorous and mathematically well-defined QFT of general validity on curved manifold enjoyed recently some renewal of interest [29, 37, 38]. However, as pointed out in the introduction, on the practical side many known caveats remain for QFT on flat spaces. They are usually taken into account either occasionally—like with the procedure of dimensional reduction [4]—or on the more fundamental level of improving in a particular way a recognized rigorous method like BPHZ [5, 30]. However, the extension of singular distributions must be envisaged both in the IR and in the UV domains, which is not the case of [5]. Here we were able to show that the concept of quantum fields defined as operator-valued distributions in flat spaces can be exploited to build a quite universal renormalization scheme if the test functions are partitions of unity defined from $\mathcal{C}^{\infty}$ functions. These two properties guarantee Lorentz and Poincaré invariances and the independence of the results from the details of the test functions. The formulation is universal in the sense that it takes care of all types of singularities which can be treated by super regular $\mathcal{C}^{\infty}$ test functions. Finite physical amplitudes are then obtained at any step of the calculations. Technically the method uses the concept of extensions of singular distributions. Interestingly enough, we have shown that it comprises usual symmetry-preserving renormalization procedures:

(i) the usual BPHZ-renormalization as a special case but also gives a precise IR analysis absent from the original BPHZ version without test functions;

(ii) Pauli–Villars-like subtractions at the level of propagators but with no need to include unphysical auxiliary fields with problematic infinite mass limits in the end;

(iii) subtracted dispersion relations at an arbitrary scale of advanced and retarded physical amplitudes.

The presented TLR scheme has already been shown to work in several applications. An important aspect of the method is that it does not violate gauge invariance. A first treatment [39] in the QED context verifies this for the polarization tensor in dimension $D = 2$ and gives Fujikawa's analysis of the axial anomaly in dimensions $D = 2, 4$ directly from the presence of the test function as partition of unity. Direct calculations at $D = 4$ of QED/QCD self-energy characteristics and Ward identities are presented in a separate publication [40]. In light-front QFT, it was shown recently that rotational invariance with respect to the light-front orientation is also preserved with TLRS ([14, 35]). Many interesting future applications may be envisaged. One may mention the incidence of TLRS on the IR catastrophe in gauge theories, on the so-called IR divergences in light-cone QFT [27, 41, 42], on chiral effective field theories, on the hierarchy problem in the standard model and in super-symmetric field theories.

Acknowledgments

We acknowledge the financial support from CNRS/IN2P3-Department during the course of this work. EW is grateful to Alain Falvard for his kind hospitality at the LPTA. We are indebted to M Slupinski (IRMA, Strasbourg) and S Baseilhac (GTA-IMM Montpellier) for numerous and valuable discussions.

Appendix A. Proof of proposition 1.

Proposition 1. *For all distribution $\Phi \in \mathcal{S}'(\mathbb{R}^n)$ and all test function $\rho \in \mathcal{S}(\mathbb{R}^n)$*

(a) the convolution product $\Phi * \rho$ is in the class $\mathcal{C}^\infty$ on $\mathbb{R}^n$,

(b) For C^∞-tempered function g of theorem 1 the application $\varphi : \mathbb{R}^n \ni x \mapsto \varphi(x) = (\Phi*\rho)(x)$ defines a function in the class $\mathcal{S}(\mathbb{R}^n : \mathbb{C})$.

Proof.

(a) (cf also chapter 6, theorem XI of [8])
$\forall x_0 \in \mathbb{R}^n |\mathrm{e}^{-\mathrm{i}\ll p,x_0\gg}\tilde{\phi}(p)f(p^2)| \leqslant |\tilde{\phi}(p)f(p^2)|$; then, from theorem 1 and the property $f \in \mathcal{S}(\mathbb{R}^n) \int_{\mathbb{R}^n} \mathrm{d}^n p|\tilde{\phi}(p)f(p^2)|$ is bounded and continuity is a simple consequence of Lebesgue's theorem on dominated convergence. In a neibourghood $\mathcal{V}$ of $x_0|\partial_x^\alpha(\mathrm{e}^{-\mathrm{i}\ll p,x\gg}\tilde{\phi}(p)f(p^2))| \leqslant F(p)$, i.e. it is a dominated derivative by an integrable function $\in \mathbb{R}^n$ $F : p \mapsto F(p)$ $\forall\alpha \in \mathbb{N}^n$, $\forall x \in \mathcal{V}(x_0)$ and for almost all $p \in \mathbb{R}^n$. Indeed, by theorem 1 and Leibnitz's rule for derivation, for all polynomial P^β $\forall\beta \in \mathbb{N}^n$ the function $P^\beta f \in \mathcal{S}(\mathbb{R}^n)$. Hence $\Phi * \rho$ is in the class $\mathcal{C}^\infty$ on $\mathbb{R}^n$.

(b) (cf also chapter 7, theorem IX of [8])
Again from theorem 1 we have $\tilde{\phi} \in \mathcal{S}'(\mathbb{R}^n)$ with $\tilde{\phi}(p) = \partial^\alpha((1 + p^2)^s g(p))$, and using Leibnitz's rule

$$
\begin{aligned}
(\Phi * \rho)(x) &= (-1)^{|\alpha|} \int \frac{\mathrm{d}^n p}{(2\pi)^n} (1 + p^2)^s g(p) \partial^\alpha \left[\mathrm{e}^{-\mathrm{i}\ll p,x\gg} f(p^2)\right] \\
&= (-1)^{|\alpha|} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-\mathrm{i})^{|\beta|} X^\beta \int \frac{\mathrm{d}^n p}{(2\pi)^n} \mathrm{e}^{-\mathrm{i}\ll p,x\gg} (1 + p^2)^s g(p) \partial^{\alpha-\beta} f(p^2)
\end{aligned}
$$

By hypothesis $g(p) \in \mathcal{C}^{\infty}$-tempered (cf definition 6), that is $h(p) = (1+p^2)^s g(p)$ and all its derivatives are at most polynomially increasing. Thus $h(p)\partial^{\alpha-\beta} f(p^2) \in \mathcal{S}(\mathbb{R}^n)$, its Fourier integral above defines a function $\rho_{(\alpha-\beta)} \in \mathcal{S}(\mathbb{R}^n : \mathbb{C}) : \mathbb{R}^n \ni x \mapsto \rho_{(\alpha-\beta)}(x)$ and the final sum on $\beta \leqslant \alpha$ does not change this feature, which establishes the assertion. □

Appendix B. Proof of theorem 3. (cf also chapter 1, theorem IV of [8])

Theorem 3. *Let $(\Omega_j)_{j\in\mathbf{J}}$ be a family of open subset of $\mathbb{R}^n$ and $(T_j)_{j\in\mathbf{J}}$ be a family of distributions such that $T_j \in \mathcal{D}'(\Omega_j)$. The family $(T_j)_{j\in\mathbf{J}}$ is supposed to fulfil the compatibility condition $T_{j|\Omega_j\cap\Omega_k} = T_{k|\Omega_j\cap\Omega_k}$ for all $j, k \in \mathbf{J}$. Then there exists a single distribution T on $\Omega = \bigcup_{j\in\mathbf{J}} \Omega_j$ such that the restriction of T to each Ω_j is T_j.*

Preamble 1. Restriction of distributions: let $\omega \subset \Omega$ be open subsets of $\mathbb{R}^n$. To any distribution $T \in \mathcal{D}'(\Omega)$ is associated its restriction, $T|_{\omega} \in \mathcal{D}'(\omega)$ such that

$$\langle T|_{\omega}, \rho\rangle = \langle T, \Theta\rangle \text{for all } \rho \in \mathcal{D}(\omega),$$

where Θ is the extension of ρ by 0 on $\Omega \setminus \omega$.

Preamble 2. If $(U_i)_{i\in I}$ is an open covering of $\mathcal{M}$, a partition of unity subordinate to $(U_i)_{i\in I}$ is a family $(\beta_j)_{j\in J}$ of positive continuous functions on $\mathcal{M}$ such that

(i) For all $X \in \mathcal{M}$ there exists a neighbourhood U_X of X such that all but a finite number of the β_j vanish on U_X.
(ii) $\sum_{i\in I} \beta_i(X) = 1$ for all $X \in \mathcal{M}$.
(iii) For all $j \in J$, there exists $i \in I$ such that the closure of $\{X \in \mathcal{M} : \beta_j(X) \neq 0\}$ is contained in U_i. The closure of $\{X \in \mathcal{M} : \beta_j(X) \neq 0\}$ is called the support of β_j.

Proposition 3. *If $(\alpha_i)_{i\in I}$ and $(\beta_j)_{j\in J}$ are two partitions of unity subordinate to $(U_k)_{k\in K}$ and $(V_l)_{l\in L}$ respectively, $(\alpha_i\beta_j)_{(i,j)\in I\times J}$ is a partition of unity subordinate to $(U_k \cap V_l)_{(i,j)\in K\times L}$.*

(1) Clearly $(U_k \cap V_l)_{(i,j)\in K\times L}$ is a covering of $\mathcal{M}$.
(2) For $X \in \mathcal{M}$ one chooses a neighbourhood U_X (resp. V_X) of X such that all but a finite number of α_i (resp. β_j) are zero on U_X (resp. V_X). Then all but a finite number of the products $\alpha_i\beta_j$ are zero on the neighbourhood $U_X \cap V_X$ of X. This proves condition (i) above for the family of functions $(\alpha_i\beta_j)_{(i,j)\in I\times J}$.
(3) Let $\alpha_1, \ldots, \alpha_A$ et $\beta_1, \ldots, \beta_B$ be the α_i (resp. β_j) which does not vanish on U_X (resp. V_X). One has then for all $x \in U_X$,

$$\alpha_1(x) + \ldots \alpha_A(x) = 1$$

and for all $x \in V_X$,

$$\beta_1(x) + \ldots \beta_A(x) = 1$$

which implies that for all $x \in U_X \cap V_X$,

$$(\alpha_1(x) + \ldots \alpha_A(x))(\beta_1(x) + \ldots \beta_A(x)) = 1.$$

The products $\alpha_i\beta_j$ appearing in this equation are the only one which does not vanish on $U_X \cap V_X$, which implies that for all $x \in U_X \cap V_X$, one has

$$\sum_{(i,j)\in I\times J} \alpha_i(x)\beta_j(x) = 1.$$

This proves condition (ii) above for the family of functions $(\alpha_i\beta_j)_{(i,j)\in I\times J}$.

(4) Since the support of a product of functions is the intersection of the supports, if for $i \in I$ one chooses $f(i) \in K$ such that $\mathrm{supp}(\alpha_i) \subseteq U_{f(i)}$ and for $j \in J$ one chooses $g(j) \in L$ such that $\mathrm{supp}(\beta_j) \subseteq V_{g(j)}$, it will hold that $\mathrm{supp}(\alpha_i\beta_j) \subseteq U_{f(i)} \cap V_{g(j)}$.
This proves condition (iii) above for the family of functions $(\alpha_i\beta_j)_{(i,j)\in I\times J}$.

Proof. Let K be a compact of Ω. A finite part $\mathbf{I} \subset \mathbf{J}$ exists such that $K \subset \bigcup_{j\in\mathbf{I}} \Omega_j$. Then a finite family $(\varphi_j)_{j\in\mathbf{I}}$ exists such that $\varphi_j \in \mathcal{D}(\Omega_j)$ and such that $0 \leqslant \varphi_j \leqslant 1$ and $\sum_{j\in\mathbf{I}} \varphi_j = 1$ in the neighbourhood of K (partition of unity or PU). If $\varphi \in \mathcal{D}_K(\Omega)$, one sets

$$\langle T, \varphi\rangle := \sum_{j\in\mathbf{I}} \langle T_j, \varphi\varphi_j\rangle$$

Notice first that T is independant of the choice of the family $\{\varphi_j\}$. Indeed if $\mathbf{I}'$ is another finite part of $\mathbf{J}$ such that $K \subset \bigcup_{k\in\mathbf{I}'} \Omega_k$ and if $\psi_k \in \mathcal{D}(\Omega_k)$ for $k \in \mathbf{I}'$ is a finite family such that $0 \leqslant \psi_k \leqslant 1$ and $\sum_{k\in\mathbf{I}'}\psi_k = 1$ in the neighbourhood of K, then

$$\begin{aligned}\sum_{j\in\mathbf{I}} \langle T_j, \varphi\varphi_j\rangle &= \sum_{j\in\mathbf{I}} \left\langle T_j, \varphi \sum_{k\in\mathbf{I}'} \varphi_j\psi_k \right\rangle = \sum_{j\in\mathbf{I}}\sum_{k\in\mathbf{I}'} \langle T_j, \varphi\varphi_j\psi_k\rangle \\ &\equiv \sum_{l\in\mathbf{L}} \langle T_l, \varphi\alpha_l\rangle,\end{aligned}$$

for the family $(\varphi_j\Psi_k)_{(j,k)\in J\times I'}$ forms an equivalent PU $(\alpha_l)_{l\in L}$.

On the other hand, $T : \mathcal{D}_K(\Omega) \mapsto \mathcal{D}_K(\Omega)$ is continuous, hence $T \in \mathcal{D}'_K(\Omega)$. It is clear that the restriction of T to each Ω_j is precisely T_j and the uniqueness of T proceeds from its definition. □

Appendix C. Loop-contributions with PU test functions

We consider first the one-loop contribution $I(k^2)$ to the four-point function of Φ^4 scalar field theory at $D = 4$. We shall work in Euclidean metric to simplify the derivation. $I(k^2)$ reads

$$\begin{aligned}I(k^2) &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{f^2(p^2) f^2((p+k)^2)}{(p^2+m^2)((p+k)^2+m^2)}, \\ &= \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{f^2((q+xk)^2)) f^2((q-(1-x)k)^2)}{(q^2+k^2x(1-x)+m^2)^2}.\end{aligned} \tag{C.1}$$

The two cases $k^2 = 0$ and $k^2 \neq 0$ must be distinguished. In the first case, with $X = \frac{p^2}{\Lambda^2}$, one has

$$I(0) = \frac{\Lambda^4}{(4\pi)^2} \int_0^\infty \frac{X\mathrm{d}X f^4(X)}{(X\Lambda^2+m^2)^2}. \tag{C.2}$$

With $T(X) = \frac{X}{(X\Lambda^2+m^2)^2}$, $\lim_{X\to\infty} T(X) \sim \frac{1}{X}$. We have then

$$\widetilde{T}(X) = \partial_X \left[\frac{X^2}{(X\Lambda^2+m^2)^2}\right] \int_1^{\eta^2} \frac{\mathrm{d}t}{t} = \frac{2Xm^2}{(X\Lambda^2+m^2)^3} \log(\eta^2). \tag{C.3}$$

It follows that

$$I(0) = \frac{2m^2\Lambda^4}{(4\pi)^2} \int_0^\infty \frac{X\mathrm{d}X}{(X\Lambda^2+m^2)^3} \log(\eta^2) = \frac{1}{(4\pi)^2} \log(\eta^2). \tag{C.4}$$

In the second case, $k^2 \neq 0$, since the test function is unity for small values of its argument, in the UV regime, $x\|k\|$ and $(1-x)\|k\|$ can be disregarded with respect to $\|q\|$ and the regulation of $I(k^2)$ is provided by $f^4(q^2)$ only. $I(k^2)$ transforms to

$$I(k^2) = \frac{\Lambda^4}{(4\pi)^2 m^4}\int_0^1 \mathrm{d}x \int_0^\infty \mathrm{d}X \frac{X f^4(X)}{\left(X\frac{\Lambda^2}{m^2} + \frac{k^2}{m^2}x(1-x)+1\right)^2},$$
$$= \frac{\Lambda^4}{(4\pi)^2 m^4}\int_0^1 \mathrm{d}x \int_0^\infty \mathrm{d}Y \frac{Y f^4\left[Y\left(\frac{k^2}{m^2}x(1-x)+1\right)\right]}{\left(Y\frac{\Lambda^2}{m^2}+1\right)^2}, \tag{C.5}$$

where, in the last equation, X has been changed to $Y\left(\frac{k^2}{m^2}x(1-x)+1\right)$. As before, we have now

$$I(k^2) = \frac{2\Lambda^4}{(4\pi)^2 m^4}\int_0^1 \mathrm{d}x \int_0^\infty \frac{Y\mathrm{d}Y}{\left(Y\frac{\Lambda^2}{m^2}+1\right)^3}\int_1^\infty \frac{\mathrm{d}t}{t} f^4\left[Yt\left(\frac{k^2}{m^2}x(1-x)+1\right)\right]. \tag{C.6}$$

As discussed in the text, the test function effectively cuts the t-integration such that $t \leqslant \frac{\eta^2}{(\frac{k^2}{m^2}x(1-x)+1)}$ in the limit $\alpha \to 1$, where the test function tends to unity over the whole integration domain in Y. Hence,

$$I(k^2) = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}x \int_1^{\frac{\eta^2}{(\frac{k^2}{m^2}x(1-x)+1)}} \frac{\mathrm{d}t}{t}$$
$$= \frac{1}{(4\pi)^2}\left\{\log(\eta^2) - \int_0^1 \mathrm{d}x \log\left[\frac{k^2}{m^2}x(1-x)+1\right]\right\}. \tag{C.7}$$

From the analysis in dimension $D = 4-\epsilon$, the result is known to be [34]

$$I(k^2) = \frac{\Gamma(\epsilon/2)}{(4\pi)^2}\left[\frac{4\pi\mu^2}{m^2}\right]^{\frac{\epsilon}{2}}\int_0^1 \mathrm{d}x\left[\frac{k^2}{m^2}x(1-x)+1\right]^{-\epsilon/2}$$
$$= \frac{1}{(4\pi)^2}\left\{\frac{2}{\epsilon} + \log\left[\frac{4\pi\mu^2 exp(-\gamma)}{m^2}\right] - \int_0^1 \mathrm{d}x \log\left[\frac{k^2}{m^2}x(1-x)+1\right]\right\}, \tag{C.8}$$

where μ is the arbitrary mass scale introduced in relation with the dimensionality of the coupling at $D = 4-\epsilon$. As expected, the divergent contribution in $\frac{2}{\epsilon}$ present in the conventional formulation has been removed in (C.7). Only the scale dependence originating from the overlapping domains building up the partition of unity remains, making clear the link between η and the arbitrary mass scale μ of dimensional regularization.

Instead of the derivation retained here, one may also use Schwinger's representation for the propagators. The test functions in this case just provide the necessary handling of divergences of the final integrals over Schwinger's parameter, with the very same result of equation (C.7). The way the regularization works is most simply seen for the propagator $\Delta(x-y)$ at $D = 4$ and $x = y$. One has

$$\Delta(0) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{f(q^2)}{q^2+m^2} \tag{C.9}$$

$$= \frac{\Lambda^4}{(4\pi)^2 m^2}\int_0^\infty y\,\mathrm{d}y\,\mathrm{e}^{-y\frac{\Lambda^2}{m^2}}\int_0^\infty \frac{\mathrm{d}u}{u^2}\mathrm{e}^{-u} f\left(\frac{y}{u}\right). \tag{C.10}$$

Now for a SRTF, it holds that

$$f\left(\frac{y}{u}\right) = -\int_1^\infty \mathrm{d}t(1-t)\partial_t^2\left[t f\left(\frac{yt}{u}\right)\right]$$
$$= -u^2\int_1^\infty \frac{\mathrm{d}t}{t}(1-t)\partial_u^2 f\left(\frac{yt}{u}\right). \tag{C.11}$$

Hence, the test function takes care of the divergence at $u = 0$, and after partial integrations, all integrals are now finite giving the expected result [43].

We exemplify now the general handling of loop contributions with Schwinger's parametrization for the two-point function $\Gamma_2[\mathbf{k}]$ with two loops. It reads

$$\begin{aligned}\Gamma_2[\mathbf{k}] &= \int \frac{d^4q_1 d^4q_2}{(2\pi)^8} \frac{f^2(q_1^2) f^2(q_2^2) f^2((q_1+q_2-k)^2)}{(q_1^2+m^2)(q_2^2+m^2)((q_1+q_2-k)^2+m^2)} \\ &= \int \frac{d^4q_1 d^4q_2 d^4q_3}{(2\pi)^8} \frac{\delta(q_3+k-q_1-q_2) f^2(q_1^2) f^2(q_2^2) f^2(q_3^2)}{(q_1^2+m^2)(q_2^2+m^2)(q_3^2+m^2)} \\ &= \int d^4y\, e^{ik\cdot y} \prod_{i=1}^{3} \int \frac{d^4q_i}{(2\pi)^4} e^{-iq_i\cdot y} \frac{f^2(q_i^2)}{(q_i^2+m^2)} \\ &= \int d^4y\, e^{ik.y} \prod_{i=1}^{3} \int d\alpha_i\, e^{-\alpha_i m^2} \int \frac{d^4q_i}{(2\pi)^4} e^{-\alpha_i q_i^2 - iq_i.y} f^2(q_i^2).\end{aligned} \tag{C.12}$$

In the last relation, $f^2(q_i^2)$ depends in fact only upon the argument $X_i = \frac{q_i^2}{\Lambda^2}$. Then with the change of integration variable $\alpha_i q_i^2 \to \widetilde{q}_i^{\,2}$, the argument of the test function becomes $\frac{\widetilde{q}_i^{\,2}}{\alpha_i \Lambda^2}$ which implies that $\alpha_i \Lambda^2 > \frac{1}{\eta^2}$. The integrals over α_i would have a lower boundary cut-off at $\frac{1}{\eta^2\Lambda^2}$, just as in the conventional treatment [43]. However, using instead Lagrange's formula for the test functions gives a finite result, whose dependence on the scale parameter η can be inferred from the conventional analysis with the replacement $\Lambda \to \eta m$.

Appendix D. Calculations leading to $\widetilde{T}(X)$, in equation (4.16)

The expression we have to evaluate first is

$$\begin{aligned}\mathcal{I}(\|X\|) &= \int_{\eta\|X\|}^{1} dt \frac{(1-t)^k}{t} \\ &= -\log(\eta\|X\|) + \sum_{p=1}^{k} (-1)^p \binom{k}{p} \int_{\eta\|X\|}^{1} dt t^{(p-1)} \\ &= -\left[\log(\eta\|X\|) + H_k + \sum_{p=1}^{k} \frac{(-1)^p}{p} \binom{k}{p} (\eta\|X\|)^p\right],\end{aligned} \tag{D.1}$$

where $H_k = \sum_{p=1}^{k} \frac{(-1)^{(p+1)}}{p} \binom{k}{p} = \gamma + \psi(k+1)$. Thus, $\widetilde{T}^{<}(X)$ is given by

$$\begin{aligned}\widetilde{T}^{<}(X) &= (-1)^{(k+1)}(k+1) \sum_{|\alpha|=k+1} \partial_X^\alpha \left[\frac{X^\alpha}{\alpha!} T(X) \int_{\eta\|X\|}^{1} dt \frac{(1-t)^k}{t}\right] \\ &= (-1)^{(k)}(k+1) \sum_{|\alpha|=k+1} \partial_X^\alpha \left\{\frac{X^\alpha}{\alpha!} T(X) \left[\log(\eta\|X\|) \right.\right. \\ &\quad \left.\left. + H_k + \sum_{p=1}^{k} \frac{(-1)^p}{p} \binom{k}{p} (\eta\|X\|)^p\right]\right\}.\end{aligned} \tag{D.2}$$

Any singular homogeneous distribution can be parametrized as $T(X) = \sum_{i=1}^{d} \frac{B(i)}{X_i^{(k+d)}}$, where the $B(i)'$s are arbitrary constant coefficients. A direct calculation then shows that

$$\sum_{|\alpha|=k+1} \partial_X^\alpha \left[\frac{X^\alpha}{\alpha!} T(X) \sum_{p=1}^{k} \frac{(-1)^p}{p} \binom{k}{p} (\eta\|X\|)^p\right] = 0.$$

Hence, (D.2) is reduced to

$$\widetilde{T}^{<}(X) = (-1)^{(k)}(k+1) \sum_{|\alpha|=k+1} \partial_X^\alpha \left\{ \frac{X^\alpha}{\alpha!} T(X)[\log(\eta\|X\|) + H_k] \right\}. \tag{D.3}$$

For a homogeneous distribution parametrized as above, one has the result [44]

$$(-1)^{(k)}(k+1) \sum_{|\alpha|=k+1} \partial_X^\alpha \left[\frac{X^\alpha}{\alpha!} T(X) \right] = \sum_{|\alpha|=k} \frac{(-1)^{|\alpha|}}{\alpha!} C^\alpha \delta^{(\alpha)}(X), \tag{D.4}$$

with $C^\alpha = \int_{(\|X\|=1)} T(X) X^\alpha \mathrm{d}S$.

This establishes the expression given in equation (4.16).

Appendix E. Example of a calculation of an UV extension in Minkowski metric

As a matter of illustration, we recalculate the (Euclidean) result of equation (C.9) in Minkowski's metric (example of integrable pole contribution). We want to give a meaning to the diverging integral $\Delta(0)$ at $D = 2, 4$:

$$\begin{aligned} \Delta(0) &= \int \mathrm{d}^D p \frac{f^2\left(p_0^2, \vec{p}^2\right)}{p_0^2 - p^2 - m^2 + 2\mathrm{i}\epsilon\omega_p} \\ &= -\int \frac{\mathrm{d}^{(D-1)} p}{2\omega_p} \int_{-\infty}^{\infty} \mathrm{d}p_0 \left[\frac{1}{\omega_p - p_0 - \mathrm{i}\epsilon} + \frac{1}{\omega_p + p_0 - \mathrm{i}\epsilon} \right] f^2\left(p_0^2, \vec{p}^2\right) \end{aligned} \tag{E.1}$$

The p_0-integration cannot be done using contour integration because the extension of the test function to the whole complex plane is not possible in general. However, one can proceed as follows. One has

$$\begin{aligned} PP \int_{-\infty}^{\infty} \mathrm{d}p_0 \frac{f^2\left(p_0^2, \vec{p}^2\right)}{p_0 \pm \omega_p} &= \lim_{\epsilon \to 0} \left\{ \int_{-\infty}^{\mp\omega_p - \epsilon} \mathrm{d}p_0 + \int_{\mp\omega_p + \epsilon}^{\infty} \mathrm{d}p_0 \right\} \\ &\quad \times \frac{1}{p_0 \pm \omega_p} \left[-p_0 \frac{\mathrm{d}}{\mathrm{d}p_0} \int_1^\infty \frac{\mathrm{d}t}{t} f^2\left(p_0^2 t^2, \vec{p}^2\right) \right] \\ &= \lim_{\epsilon \to 0} \left[\pm \frac{\omega_p}{\epsilon} - 1 \mp \frac{\omega_p}{\epsilon} \pm \frac{\omega_p}{\epsilon} + 1 \mp \frac{\omega_p}{\epsilon} \right] \log[\eta^2] = 0, \end{aligned} \tag{E.2}$$

where a partial integration on p_0 has been performed. Hence,

$$\int_{-\infty}^{\infty} \mathrm{d}p_0 \frac{f^2\left(p_0^2, \vec{p}^2\right)}{p_0 \pm \omega_p \mp i\epsilon} = \pm \mathrm{i}\pi f^2\left(\omega_p^2, \vec{p}^2\right). \tag{E.3}$$

At dimension $D = 2$, one obtains

$$\begin{aligned} \Delta(0) &= -\mathrm{i}\pi \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{\omega_p} f^2\left(\omega_p^2, \vec{p}^2\right), \\ &= -\mathrm{i}\pi \left\langle \frac{1}{\omega_p}, f^2\left(\omega_p^2, \vec{p}^2\right) \right\rangle = -\mathrm{i}\pi \left\langle \widetilde{\frac{1}{\omega_p}}, 1 \right\rangle. \end{aligned} \tag{E.4}$$

Then

$$\begin{aligned} \widetilde{\left(\frac{1}{\omega_p}\right)} &= \frac{\mathrm{d}}{\mathrm{d}p} \left[p \int_1^{\eta^2} \frac{\mathrm{d}t}{t} \frac{1}{\sqrt{p^2 + m^2 t^2}} \right], \\ &= \frac{1}{\sqrt{p^2 + m^2}} - \frac{1}{\sqrt{p^2 + m^2\eta^4}}. \end{aligned} \tag{E.5}$$

The end result for $\Delta(0)$ scales then as $\log[\eta^2]$ as expected. The two contributions in equation (E.5) may be recombined to give a PV-type of subtraction at the level of the propagator:

$$\Delta(0) = \int \mathrm{d}^2 p \frac{f^2(p_0^2, \vec{p}^2)}{p^2 - m^2 + \mathrm{i}\epsilon},$$
$$= \int \mathrm{d}^2 p \left[\frac{1}{p^2 - m^2 + \mathrm{i}\epsilon} - \frac{1}{p^2 - m^2\eta^4 + \mathrm{i}\epsilon} \right]. \tag{E.6}$$

It is important to note that the non-causal light-cone singularity in $\delta(x^2)$ now present in each contribution of equation (E.6) just cancels out in the subtraction. In EG causal perturbation theory, it is a general feature that—by definition—diverging causality-violating terms are avoided right from the beginning [32]. At dimension $D = 4$, one has

$$\Delta(0) = -4\mathrm{i}\pi^2\Lambda^3 \int_0^\infty \frac{X \mathrm{d}X f^2(X)}{\sqrt{X(X\Lambda^2 + m^2)}},$$
$$= -4\mathrm{i}\pi^2\Lambda^3 \left\langle \widetilde{\frac{1}{\sqrt{X(X\Lambda^2 + m^2)}}}, 1 \right\rangle. \tag{E.7}$$

Thus,

$$\left(\widetilde{\frac{1}{\sqrt{X(X\Lambda^2 + m^2)}}} \right) = -\partial_X^2 \left[X^2 \int_1^{\eta^2} \mathrm{d}t \frac{(1-t)}{t^2} \frac{1}{\sqrt{X(X\Lambda^2 + m^2 t)}} \right]. \tag{E.8}$$

The final t and X integrations give back the known result [43].

References

[1] Bogoliubov N N and Parasiuk O S 1957 *Acta Math.* **97** 227
Bogoliubov N N and Shirkov D V 1980 *Introduction to the Theory of Quantized Fields* 3rd edn (New York: Wiley)

[2] Zimmerman W 1972 *Ann. Phys., NY* **77** 536

[3] ’t Hooft G and Veltman M 1972 *Nucl. Phys.* B **44** 189

[4] Capper D M, Jones D R T and van Nieuwenhuizen P 1980 *Nucl. Phys.* B **167** 479

[5] Falk J, Häußling R and Scheck F 2010 *J. Phys. A: Math. Theor.* **43** 035401

[6] Epstein H and Glaser V 1973 *Ann. Inst. Henri Poincaré* **XIXA** 211

[7] Haag R 1996 *Local Quantum Physics: Fields, Particules, Algebras (Texts and Monographs in Physics)* 2nd edn (Berlin: Springer)

[8] Schwartz L 1966 *Théorie des distributions* (Paris: Hermann)

[9] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* vol 1 *(Tracts in Pure and Applied Mathematics* vol 15) (New York: Interscience)

[10] Hörmander L 1969 *Linear Partial Differential Equation* (Berlin: Springer)

[11] Hörmander L 1986 *The Analysis of Linear Partial Differential Operators* (Berlin: Springer)

[12] Prange J 1999 *J. Phys. A: Math. Gen.* **32** 2225

[13] Brunetti R and Fredenhagen K 2000 Microlocal analysis and interacting field theories: renormalization on physical background *Commun. Math. Phys.* **208** 623 (arXiv:math-ph/990328)

[14] Grangé P, Mathiot J F, Mutet B and Werner E 2010 *Phys. Rev.* D **82** 025012

[15] Sobolev S L 1991 *Some Applications of Functional Analysis in Mathematical Physics* (Providence, RI: American Mathematical Society)

[16] Gelfand I M and Shilov G E 1964 *Generalized Functions* vols 1–3 (New York: Academic)

[17] Lützen J 1982 *The Prehistory of the Theory of Distributions* (New York: Springer)

[18] Jaffe A and Ritter G 2007 Quantum field theory on curved backgrounds I and II *Comm. Math. Phys.* **270** 545

[19] Golse F 2010 Distributions, analyse de Fourier équations aux dérivées partiellers *Cours Ecole Polytechnique* (Paris)

[20] Schweber S S 1964 *An Introduction to Relativistic Quantum Field Theory* (New York: Harper and Row) p 721
Collins J 1984 *Renormalization* (Cambridge: Cambridge University Press) p 4

[21] Streater R F and Wightman A S 1964 *PCT, Spin & Statistics, and All That* (New York: Benjamin)
[22] Dieudonné J 1944 Une Généralisation des Espaces Compacts *Math. Pure Appl.* **23** 65
[23] Stone A H 1948 Paracompactness and product spaces *Bull. Am. Math. Soc.* **54** 977
[24] Spivak M 1979 *A Comprehensive Introduction to Differential Geometry* (Waltham, MA: Brandeis University)
Nakahara M 1990 *Geometry, Topology and Physics (Graduate Student Series in Physics)* (Bristol: Institute of Physics Publishing)
[25] Osterwalder K and Schrader 1973 Axioms for Euclidean Green's functions *Comm. Math. Phys.* **31** 83
[26] Reed M and Simon B 1975 *Methods of Modern Mathematical Physics II* (London: Academic)
[27] Ullrich P 2004 *J. Math. Phys.* **45** 3109
[28] Bogoliubov N N, Logunov A A, Oksak A I and Todorov L T 1990 *General Principles of Quantum Mechanics* (Dordrecht: Kluwer)
[29] Kuznetsov A N, Tkachov A V and Vlasov V V 1996 arXiv:hep-th/9612037
Pinter G 2001 *Ann. Phys.* 8 **10** 333
[30] Gracia-Bondia J M 2003 *Math. Phys. Anal. Geom.* **6** 59
Gracia-Bondia J M and Lazzarini S 2003 *J. Math. Phys.* **44** 3863
[31] Grangé P and Werner E 2006 *Nucl. Phys. (Proc. Suppl.)* B **161** 75
[32] Scharf G 1995 *Finite QED: The Causal Approach* (Berlin: Springer)
[33] John Collins J, Perez A and Sudarsky D 2009 Lorentz invariance violation and its role in quantum gravity phenomenology, arXiv:hep-th/0903002
John Collins J, Perez A and Sudarsky D 2009 *Quantum Gravity* ed D Oriti (Cambridge: Cambridge University Press)
[34] Itzykson C and Zuber J B 1980 *Quantum Field Theory* (New York: McGraw-Hill)
[35] Grangé P, Mathiot J F, Mutet B and Werner E 2009 *Phys. Rev.* D **80** 105012
[36] Grangé P, Mutet B and Werner E 2008 *PoS(LC2008)* 005 (available at http://pos.sissa.it/archive/conferences/061/005/LC2008_005.pdf)
[37] Brunetti R and Fredenhagen K 1997 Interacting quantum fields in curved space: renormalizibilty of φ^4 *Operator Algebra and Quantum Field Theory* ed S Doplicher *et al* (Hong Kong: International) (arXiv:gr-qc/9701048)
[38] Bahns D, Doplicher S and Fredenhagen K 2003 *Commun. Math. Phys.* **237** 221
[39] Grangé P and Werner E 2005 *Few-Body Syst.* **35** 103
[40] Grangé P, Mutet B and Werner E 2010 *PoS(LC2010)* 026 (available at http://pos.sissa.it/archive/conferences/119/026/LC2010_026.pdf)
[41] Grangé P, Ullrich P and Werner E 1998 *Phys. Rev.* D **57** 4981
[42] Salmons S, Grangé P and Werner E 1999 *Phys. Rev.* D **60** 067701
Salmons S, Grangé P and Werner E 2002 *Phys. Rev.* D **65** 125015
[43] Itzykson C and Drouffe J M 1989 *Théorie statistique des champs* (Paris: Savoirs Actuels, InterEditions, Editions du CNRS)
[44] Estrada R and Kanwall R P 2002 *A Distributional Approach to Asymptotics: Theory and Applications (Birkhäuser Advanced Texts)* (Boston: Birkhäuser) chapter 2